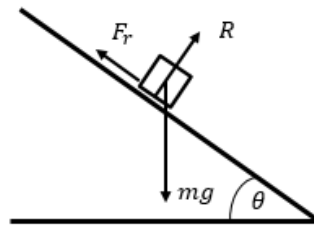# Qu 1.

a) Skier on a slope.

- Diagram with forces ✓

- Resolve down the slope $\qquad mg\sin\theta - F_r = ma$ ✓
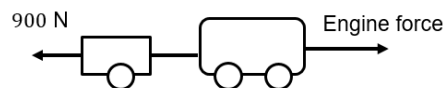
- substitute $\qquad a = \dfrac{2d}{t^2}$

- Hence $\qquad F_r = m\left(g\sin\theta - \dfrac{2d}{t^2}\right)$ ✓

(3 marks)

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b) Car & caravan.

- $ma = \text{engine force} - \text{total drag}$
  So, engine force $= ma + \text{drag}$
  $\qquad\qquad = 1800 \times 2 + 900$
  engine force $= 4500\,\text{N}$ ✓

- Tension in tow bar $= ma + 300\,\text{N}$
  $\qquad\qquad = 2 \times 600 + 300 = 1500\,\text{N}$ ✓
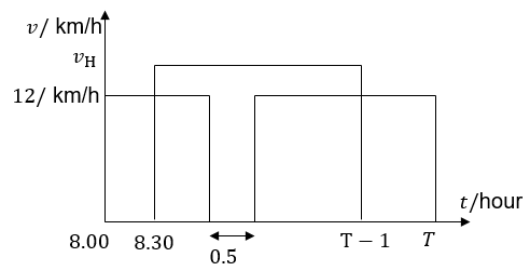
- power $= Fv = 4500\,\text{N} \times 10\,\text{m s}^{-1} = 45\,000\,\text{W}$ ✓

(3 marks)

---

c) Helen & Robert cycling.

-

- For Robert: $50\,(\text{km}) = (T - 0.5) \times 12\,(\text{km h}^{-1})$ ✓
  $\qquad\qquad T = \dfrac{56}{12}\,\text{h}$

- For Helen: $50 = (T - 1.5)v_H$ ✓
  $\qquad\qquad 50 = (56/12 - 3/2)v_H$

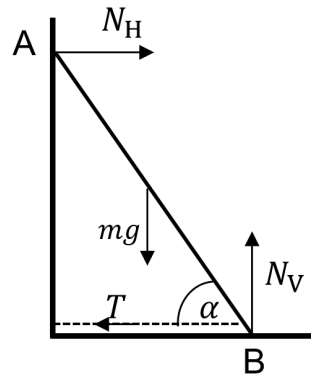- Hence, $\qquad v_H = (50 \times 38)/12 = 15.8 = 16\,\text{km h}^{-1}$ ✓

(3 marks)

---

d) Plank on wall.

-

Any variations allowed.

- Take moments about A: $\quad mg\dfrac{\ell}{2}\cos\alpha + T\ell\sin\alpha - N_{\mathrm V}\ell\cos\alpha = 0$ ✓

- Resolving vertically $\qquad\qquad mg = N_{\mathrm V}$ ✓

- Hence $\qquad\qquad \frac{mg}{2}\cos\alpha + T\sin\alpha = mg\cos\alpha$
$$T\sin\alpha = \tfrac{mg}{2}\cos\alpha$$

Therefore $\qquad\qquad\qquad\qquad T = \frac{mg}{2}\cot\alpha$ ✓

(3 marks)

---

e) Time & ratios for a falling object.

The object fall for T seconds.
Using $s = \frac{1}{2}gt^2$, we have (ℓ = last second, p = penultimate second)

- $\qquad\qquad s_\ell = \frac{1}{2}gT^2 - \frac{1}{2}g(T-1)^2$
and $\qquad s_p = \frac{1}{2}g(T-1)^2 - \frac{1}{2}g(T-2)^2$

So, $\qquad \dfrac{s_\ell}{s_p} = \dfrac{T^2 - (T-1)^2}{(T-1)^2 - (T-2)^2}$

Hence $\qquad \dfrac{s_\ell}{s_p} = \dfrac{\cancel{T^2} - \cancel{T^2} + 2T - 1}{\cancel{T^2} - 2T + 1 - \cancel{T^2} + 4T - 4}$

So that $\qquad \dfrac{2T-1}{2T-3} = \dfrac{3}{2}$

Giving $\qquad T = 7/2 = 3.5\,\mathrm{s}$ ✓

- $s = \frac{1}{2}9.8 \times 3.5^2 = 60\,\mathrm{m}$ $\qquad$ (or even 61 m using g=10) ✓

- $v = at = 9.8 \times 3.5 = 34.3 = 34\,\mathrm{m\,s^{-1}}$ ✓

(3 marks)

An alternative: distance fallen is average speed x time interval. Times increasing by 1 s are
$t_2, t_1, t_0 = T$
$s_p = \dfrac{gt_1 + gt_2}{2}(t_1 - t_2)$
$s_\ell = \dfrac{gt_1 + gt_0}{2}(t_0 - t_1)$
So $\dfrac{s_\ell}{s_p} = \dfrac{t_1 + t_0}{t_2 + t_1} = \dfrac{3}{2}$
Then $3(t_2 + t_1) = 2(t_1 + t_0)$
And using $\qquad t_0 - t_1 = 1$ and $t_1 - t_2 = 1$ we obtain $t_0 = T = 3.5\,\mathrm{s}$ as above.

4

f) Two wires from a horizontal rod.

- The weight is symmetrically supported by the two wires.
  Resolving vertically, $2T \cos \theta = mg$
  Hence $\qquad T = \dfrac{mg}{2\frac{\sqrt{3}}{2}} = \dfrac{mg}{\sqrt{3}}$ ✓

- When the wire is cut, we can resolve radially along AC.
  $T_{AC} - mg \cos 30° = ma_r$
  But $\qquad a_r = \dfrac{v^2}{r} = 0$ at this initial moment.
  Hence $\qquad T_{AC} = mg \cos 30° = mg \dfrac{\sqrt{3}}{2}$ ✓

- Thus $\qquad \dfrac{T_{AC}}{T} = \dfrac{mg \cos 30}{mg/(2\cos 30)} = 2\cos^2 30 = mg\dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{3}}{mg} = \dfrac{3}{2}$ ✓

(3 marks)

g) Long copper wire.

- Since $R = \dfrac{\rho \ell}{A} \Rightarrow \ell = \dfrac{RA}{\rho}$

- Density, d is given by $\qquad d = \dfrac{m}{V} = \dfrac{m}{A\ell} \Rightarrow A = \dfrac{m}{\ell d}$ ✓
  Hence $\qquad \ell = \dfrac{RA}{\rho} = \dfrac{Rm}{\rho \ell d}$

- So $\qquad \ell^2 = \dfrac{Rm}{\rho d}$ ✓
  Therefore $\qquad \dfrac{\ell_2{}^2}{\ell_1{}^2} = \dfrac{R_2}{R_1} \times \dfrac{m_2}{m_1}$
  $\qquad\qquad\qquad = \dfrac{6000}{0.15}\dfrac{10^3}{10^{-3}} = 4 \times 10^{10}$
  So that $\qquad \ell_2 = 2.0 \times 10^5 \, \text{m}$ ✓
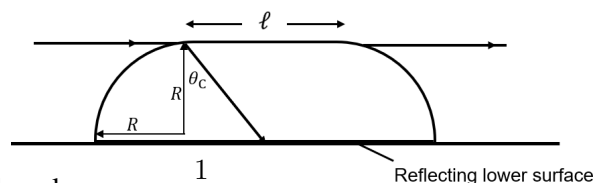
(3 marks)

h) The ray of light has to enter the glass and it follows a symmetric path.

- It must enter with the smallest ℓ value and take the steepest path down to the bottom. This is when it enter at the top of the curve and the refracted angle in the glass is the critical angle. Mark for diagram or idea of critical angle. ✓

- $\dfrac{\ell/2}{R} = \tan \theta_{\text{critical}}$ ✓

for the critical angle $n = \dfrac{1}{\sin \theta_{\text{critical}}}$

So $\qquad \dfrac{\ell/2}{R} = \dfrac{1}{n \cos \theta_{\text{critical}}}$

And $\qquad \cos \theta_{\text{critical}} = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - 1/n^2}$

Then $\qquad \ell = \dfrac{2R}{\sqrt{n^2 - 1}} = \dfrac{4}{\sqrt{1.46^2 - 1}} = 3.5(3) \, \text{cm}$ ✓

(3 marks)

5

i) Beam of photons.

- $F = \dfrac{\Delta p}{\Delta t} = \dfrac{1}{c}\dfrac{\Delta E}{\Delta t} = \dfrac{\text{Power}}{c}$ ✓
- For reflection, there is a factor of 2 for 50% of the power

  So net force $= \dfrac{3P}{2c} = \dfrac{4}{c} \times 2 + \dfrac{4}{c}$ ✓

$= \dfrac{12}{c} = \dfrac{12}{3 \times 10^8}$

$= 4 \times 10^{-8}\,\text{N}$ ✓

- In time Δt there are $N = nc\Delta t A$ photons in a cylinder of beam.

Then $n = \dfrac{N}{cA\Delta t} = \dfrac{P \times \Delta t)/hf}{cA\Delta t}$

Giving $n = \dfrac{P}{hfcA} = \dfrac{P\lambda}{hc^2 A} = \dfrac{8 \times 600 \times 10^{-9}}{6.63 \times 10^{-34} \times 9 \times 10^{16} \times 0.0012}$

$n = 6.7 \times 10^{13}\,\text{m}^{-3}$ ✓

(4 marks)

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j) neutron time of flight.

- time of flight $= \dfrac{11}{2200} = 5 \times 10^{-3}\,\text{s}$ ✓

  Using $\qquad N = N_0 \exp\left(-\lambda t\right) = N_0 \exp\left(-\ln 2 . \dfrac{t}{t_{\text{hl}}}\right)$

- With $\qquad \ln 2 . \dfrac{t}{t_{\text{hl}}} = \ln 2 \times \dfrac{5 \times 10^{-3}}{880} = 3.94 \times 10^{-6}$ ✓

  The exponent is small so an approximation is used, $e^{-x} \approx 1 - x$

  So that for $N_{\text{decays}} = N_0 - N = N_0 \left(1 - \dfrac{N}{N_0}\right)$

  $= N_0(1 - e^{-\lambda . t}) \approx N_0 \lambda t = N_0 \ln 2 \dfrac{t}{t_{\text{hl}}}$ ✓

  $= 10^8 \times 3.94 \times 10^{-6} = 394 = 390 = 400$ neutrons. ✓

  Lose a mark for missing $\ln 2$ factor, which results in 570 neutrons.

Alternatively

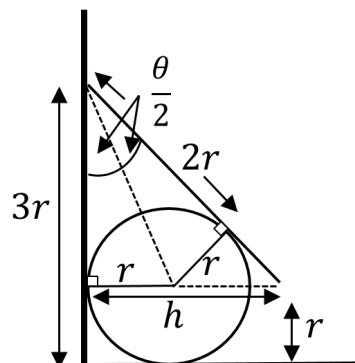Since time of flight $\ll t_{\text{hl}}$

$dN = N\lambda dt = 10^8 \times 0.693 \times \dfrac{5 \times 10^{-3}}{880} = 390$ neutrons $\qquad$ (3 marks here + 1 for ToF)

(4 marks)
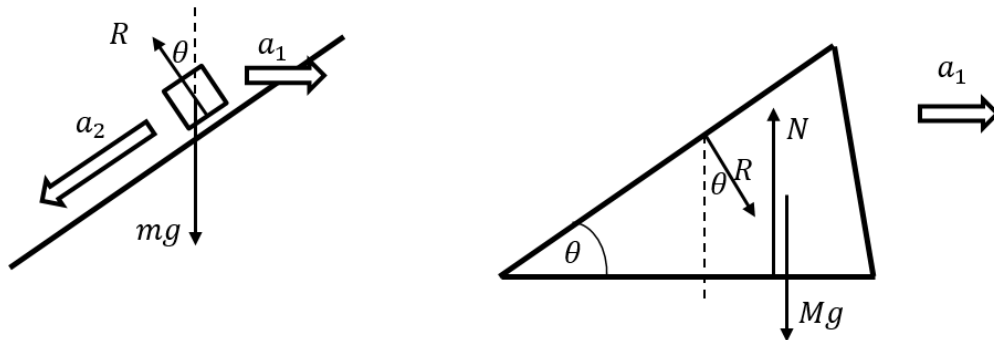
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k) Road & cylinder.

-

6

- $\tan\frac{\theta}{2} = \dfrac{r}{2r} = \dfrac{1}{2}$

  so $\quad \tan\theta = \dfrac{2.\frac{1}{2}}{1-\frac{1}{4}} = \dfrac{4}{3}$ ✓

- $\dfrac{h}{2r} = \tan\theta = \dfrac{4}{3} \quad\Rightarrow h = \dfrac{8r}{3}$

  Then $\quad \ell^2 = (2r)^2 + h^2 = 4r^2 + \dfrac{64}{9}r^2 = \dfrac{100}{9}r^2$

  So $\quad \ell = \dfrac{10}{3}r$ ✓

- The maximum amount of work is such that the tip of the rod is raised from r to $2r$.
  The CM of the rod is raised by $\frac{1}{2}r$.

  So the WD is $\frac{1}{2}mgr$ ✓

- $\cos\theta_0 = \dfrac{2r}{10r/3} = \dfrac{3}{5}$

  $\cos\theta_{\max} = \dfrac{r}{10r/3} = \dfrac{3}{10}$

  Hence $\quad \dfrac{\cos\theta_0}{\cos\theta_{\max}} = 2$ ✓

(4 marks)

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l) Mass on a sliding wedge..

- Diagram of forces ✓

  The block accelerates down the slope, whilst maintaining contact with the slope. So there are two components to its acceleration, a_2 and a_1. If we resolve perpendicular to the slope, we do not need to know a_2.
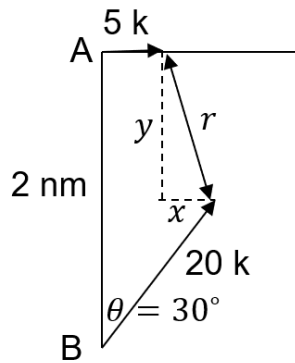
- Wedge: resolve H $\qquad\qquad R\sin\theta = Ma_1$ ✓
- particle: resolve $\perp$ to slope $\qquad mg\cos\theta - R = ma_1\sin\theta$ ✓

- Writing $\qquad\qquad\qquad R = \dfrac{Ma_1}{\sin\theta}$

  Then, $\qquad\qquad\qquad mg\cos\theta - \dfrac{Ma_1}{\sin\theta} = ma_1\sin\theta$

  Thus $\qquad\qquad mg\cos\theta = a_1\left(\dfrac{M}{\sin\theta} + m\sin\theta\right)$

  Then $\qquad\qquad mg\cos\theta\sin\theta = a_1(M + m\sin^2\theta)$

  So $\qquad\qquad\qquad a_1 = \dfrac{\frac{1}{2}mg\sin 2\theta}{(M + m\sin^2\theta)}$

  For $\theta = 30°$

  $$a_1 = \dfrac{\frac{1}{2}mg\frac{\sqrt{3}}{2}}{(M + m\frac{1}{4})} = \dfrac{\sqrt{3}mg}{(4M + m)}$$ ✓

(4 marks)

m) Two ships.

- Diagram ✓

- $y = 2 - 10\sqrt{3}t$ one mark for both the x and y equations here ✓
  $x = t(5 - 10) = -5t$

- $$r^2 = x^2 + y^2 = 25t^2 + 300t^2 + 4 - 40\sqrt{3}t$$ ✓
  which is $r^2 = 325t^2 - 40\sqrt{3}t + 4$

- Either solve quadratic in t: $325t^2 - 40\sqrt{3}t + (4 - r^2) = 0$ for zero discriminant,
  $40^2.3 - 4.325.(4 - r^2) = 0$ which gives $r = \dfrac{2}{\sqrt{13}}$, or   [✓✓]

- OR differentiate w.r.t. time $2r\dfrac{dr}{dt} = 650t - 40\sqrt{3}$

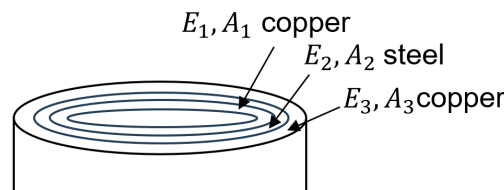 $= 0$ when $t = \dfrac{40\sqrt{3}}{650}$ ✓

- Therefore $r^2_{\min} = 325\dfrac{40^2.3}{650^2} - 40\sqrt{3}\dfrac{40\sqrt{3}}{650} + 4$

  $= \dfrac{4}{13}$

  Hence $r_{\min} = \dfrac{2}{\sqrt{13}} = 0.55\,\text{nautical miles}$ ✓

  (5 marks)

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n) Concentric cylinders.

- $E = \dfrac{F_1}{A_1}\dfrac{\ell}{\delta\ell}$

  So that $F_1 = E_1A_1\dfrac{\delta\ell}{\ell}, F_2 = E_2A_2\dfrac{\delta\ell}{\ell}, F_3 = E_3A_3\dfrac{\delta\ell}{\ell}$

  Now the applied stress $= \dfrac{\sum_{i=1}^{3}F_i}{\sum_{i=1}^{3}A_i} = \dfrac{(E_1A_1 + E_2A_2 + E_3A_3)}{(A_1 + A_2 + A_3)}\cdot\dfrac{\delta\ell}{\ell}$ ✓

  Hence $\delta\ell = \left[\dfrac{\Sigma F_i}{\Sigma A_i}\right] \times \dfrac{\Sigma A_1}{\Sigma E_i A_i} \times \ell$

  In the question the diameter was given, not the radius. Either is accepted. The diameter
  $= 4.0\,\text{cm}$ answers would be a factor of $4\times$ smaller than those here.
  $A_1 + A_3 = \pi(4.00^2 - 3.8^2) \times 10^{-4} + \pi(4.45^2 - 4.25^2) \times 10^{-4}$
  $\qquad = \pi \times (1.74 + 1.56) \times 10^{-4}$
  $\qquad = \pi \times 3.3 \times 10^{-4}\,\text{m}^2$ If $4.0\,\text{cm}$ is (correctly) used as the diameter
  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad A_1 + A_3 = \pi \times 0.825 \times 10^{-4}\,\text{m}^2$

  $A_2 = \pi(4.25^2 - 4.00^2) \times 10^{-4}$
  $\qquad = \pi \times 2.06 \times 10^{-4}\,\text{m}^2$ $A_2 = \pi \times 0.516 \times 10^{-4}\,\text{m}^2$

8

Area of copper and steel tubes
$$\sum_{i=1}^{3} A_i = \pi \times 5.36 \times 10^{-4}\,\mathrm{m}^2 \qquad {\color{red}\sum_{i=1}^{3} A_i = \pi \times 1.34 \times 10^{-4}\,\mathrm{m}^2} \qquad \checkmark$$

And $\Sigma E_i A_i = (200 \times 10^9 \times \pi \times 2.06 \times 10^{-4}) + (110 \times 10^9 \times 3.3 \times 10^{-4})$
$$= \pi \times 10^5 (200 \times 2.06 + 110 \times 3.3)$$
$$= \pi \times 7.75 \times 10^5\,\mathrm{N} \qquad\qquad {\color{red}\pi \times 1.94 \times 10^5\,\mathrm{N}}$$

So the extension is $\qquad \delta\ell = 6.2 \times 10^7 \times \dfrac{\pi \times 5.36 \times 10^{-4}}{\pi \times 7.75 \times 10^7} \times 0.30$
$$= 6.2 \times \dfrac{2 \times 5.36 \times 10^{-4} \times 0.30}{7.75}$$
$$= 1.29 \times 10^{-4}\,\mathrm{m} \qquad\qquad {\color{red}5.16 \times 10^{-4}\,\mathrm{m}} \qquad \checkmark$$

Stress in copper tubes $= E_{\mathrm{copper}}\dfrac{\delta\ell}{\ell} = 110 \times 10^9 \times \dfrac{1.29 \times 10^{-4}}{0.3} = 4.7 \times 10^7\,\mathrm{Pa} \qquad \checkmark$
$${\color{red}= 1.9 \times 10^8\,\mathrm{Pa}}$$

Total load $= (E_s A_s + E_c A_c) \times \dfrac{\delta\ell}{\ell} \qquad {\color{red}= \text{Given stress} \times \text{total area} \times \dfrac{\delta\ell}{\ell}}$
$${\color{red}= 6.4 \times 10^7 \times \pi \times 5.36 \times 10^{-4} \times \dfrac{\delta\ell}{\ell} = 1 \times 10^5\,\mathrm{N}}$$
$$= (200 \times 2.06 + 110 \times 3.3) \times 10^9 \times \pi \times 10^{-4} \times \left(\dfrac{1.29 \times 10^{-4}}{0.30}\right)$$
$$= 1 \times 10^5\,\mathrm{N} \qquad {\color{red}\text{the same value - ALLOW ecf with their total area}} \qquad \checkmark$$

(5 marks)

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o) Standing wave on a wire.
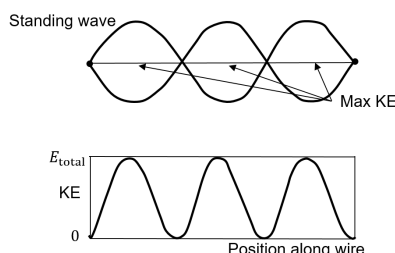
- A particle on the wire oscillates with SHM
$x = A\sin\omega t$
$v = A\omega\cos\omega t$
So the KE of a point would be given by $\frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2\omega t \qquad \checkmark$
and the KE$_\mathrm{m}$ax $= \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2$
Average value of KE for the wire requires average value of $\sin^2\theta$ over three cycles, which is $\frac{1}{2}$ $\qquad \checkmark$
So the KE of the wire will be $\frac{1}{4}M_{\mathrm{wire}}A\omega^2 \qquad \checkmark$

Given the average KE and PE are equal, the energy in the wire is KE+PE $= \frac{1}{2}M_{\mathrm{wire}}A\omega^2$
To obtain ω we need the speed of the wave and its wavelength:
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{45}{6.0 \times 10^{-3}/1.2}} = \sqrt{9 \times 10^3} = 94.87 = 95\,\mathrm{m\,s}^{-1}$$
and $f = \dfrac{v}{\lambda} = \dfrac{94.97}{1.2/3 \times 2} = \dfrac{94.87}{0.8} = 118.6 = 120\,\mathrm{Hz} \qquad \checkmark$
So $E_{\mathrm{total}} = \frac{1}{2}M_{\mathrm{wire}}A^2\omega^2$
$$= 0.5 \times 6 \times 10^{-3} \times (1.8 \times 10^{-2})^2 \times 4\pi^2 \times 118.6^2 = 0.54\,\mathrm{J} \qquad \checkmark$$

(5 marks)

9

p) Maximum power.

- Currents I, I_1, I_2 through the cell, R_1 and R_2

$$I = \frac{\varepsilon}{r + \dfrac{R_1 R_2}{(R_1 + R_2)}} = \frac{\varepsilon(R_1 + R_2)}{r(R_1 + R_2) + R_1 R_2}$$ ✓

- Potential across each resistor is the same, so $I_1 R_1 = I_2 R_2$

and $I_1 + I_2 = I$

So $I_2 R_2 = (I - I_2)R_1$

$$I_2 = \left(\frac{R_2}{R_1} + 1\right) = I$$

Hence $I_2 = \dfrac{I R_1}{(R_1 + R_2)}$ ✓

- $P_2 = I_2^2 R_2 = \dfrac{I^2 R_1^2}{(R_1 + R_2)^2} R_2$

$$= \frac{\varepsilon^2 R_1^2 R_2}{[r(R_1 + R_2) + R_1 R_2]^2}$$ ✓

- Max P_2 is given by min $\dfrac{1}{P_2} = \dfrac{1}{\varepsilon^2 R_1^2} \dfrac{[r(R_1 + R_2) + R_1 R_2]^2}{R_2}$

$$= \frac{1}{\varepsilon^2 R_1^2} \frac{[rR_1 + R_2(r + R_1)]^2}{R_2}$$

$$= \frac{1}{\varepsilon^2 R_1^2} \left[\frac{r^2 R_1^2}{R_2} + \frac{R_2^2(r + R_1)^2}{R_2} + \frac{2r R_1 R_2(r + R_1)}{R_2}\right]$$

For the term in the square brackets $[\,]$ we can differentiate with respect to R_2 and set this equal to zero.

So we only need to consider the terms with R_2, which are $\dfrac{r^2 R_1^2}{R_2} + R_2(r + R_1)^2$

And differentiating and setting to zero, we obtain $0 = -\dfrac{r^2 R_1^2}{R_2^2 + (r + R_1)^2}$

We can separate the two terms and take the square root, which gives $R_2 = \dfrac{r R_1}{(r + R_1)}$ ✓✓

or $$\frac{1}{R_2} = \frac{1}{r} + \frac{1}{R_1}$$

(5 marks)

---

q) Capacitor circuit.

- $V_{1,2,3}$ across $C_{1,2,3}$ respectively.

$$
\begin{array}{llc}
 & E_1 = V_1 + V_3 & (1) \\
 & E_2 = V_2 + V_3 & (2) \\
\text{So} & E_1 - E_2 = V_1 - V_2 & (3) \\
\text{Charge conserved} & Q_3 = Q_1 + Q_2 & (4) \\
Q = CV & V_3 C_3 = V_1 C_1 + V_2 C_2 & (5)
\end{array}
$$

✓
✓

✓

- Add $(1) + (2)$ $E_1 + E_2 = V_1 + V_2 + 2V_3$ (6)

Eliminate V_3 using (3)
$$E_1C_3 + E_2C_3 - V_2C_3 - V_2C_3 = 2V_1C_1 + 2V_2C_2$$

tidying
$$C_3(E_1 + E_2) = V_1(2C_1 + C_3) + V_2(2C_2 + C_3)$$

Eliminate V_1 using (3)
$$C_3(E_1 + E_2) = (E_1 - E_2 + V_2)(2C_1 + C_3) + V_2(2C_2 + C_3)$$

So
$$C_3(E_1 + E_2) = (E_1 - E_2)(2C_1 + C_3) + V_2(2C_1 + 2C_2 + 2C_3)$$

then
$$\cancel{E_1 C_3} + E_2C_3 = E_1C_1 - E_22C_1 + \cancel{E_1 C_3} - E_2C_3 + 2V_2.\Sigma C_i)$$

Divide through by 2
$$E_2C_3 = E_1C_1 - E_2C_1 + V_2\Sigma C_i$$

Hence
$$V_2 = \frac{E_2(C_1 + C_3) - E_1C_1}{\Sigma C_i}$$

Changing the indices
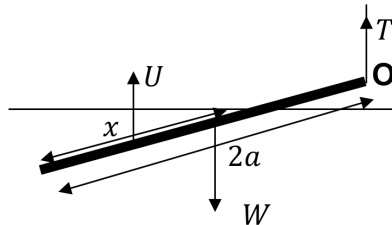$$V_1 = \frac{E_1(C_2 + C_3) - E_2C_2}{\Sigma C_i}$$

Using (1), $V_3 = E_1 - V_1 = V_3 = \dfrac{E_1\Sigma C_i - E_1C_2E_1C_3 + E_2C_2}{\Sigma C_i}$

$$V_3 = \frac{E_1C_1 + E_2C_2}{(C_1 + C_2 + C_3)}$$ ✓✓

(5 marks)

r) Floating rod.

- Cross sectional area is A. x is the length submerged
- The upthrust acts at a point halfway along the submerged length at $(2a - \frac{x}{2})$ from O.   ✓
  Weight $W = \rho A 2ag$
  Upthrust $U = \frac{4}{3}\rho Axg$ ✓

- Moments about O: $(2a - \frac{x}{2}).\frac{4}{3}\rho Axg = a.\rho A2ag$ ✓
  giving $(2a - \frac{x}{2})\frac{4}{3}x = 2a^2$
  Hence $8ax - 2x^2 = 6a^2$ or $x^2 - 4ax + 3a^2 = 0$
  Solving $x = 2a \pm a$ from which $x = a$, or fraction submerged $= \dfrac{1}{2}$ ✓

- Taking moments about the centre of the rod we can obtain the tension T ✓
  $$a.T = (a - \tfrac{x}{2}).\tfrac{4}{3}\rho A\frac{a}{2}g$$
  With $x = a$
  $$a.T = \tfrac{4}{3}\rho A\frac{a}{2}g$$
  $$T = \tfrac{2}{3}\rho Ag \quad \text{or} \quad \left(\frac{T}{W} = \tfrac{1}{3}\right)$$ ✓

(6 marks)

s) Two balls above a beaker of water.

- Two forces on each ball and a tension in the string (labelled in words is fine) $\Rightarrow\Rightarrow$ ✓
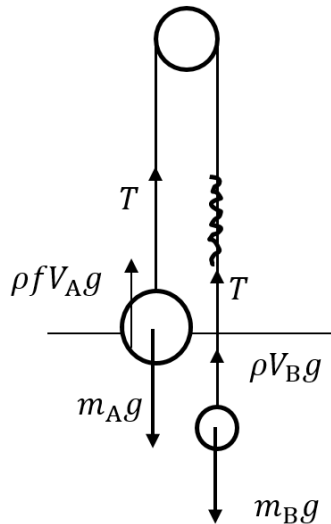
- By Archimedes we can equate forces on the string. f is the fraction of the volume of A submerged

- $$T = m_{\mathrm{B}}g - \rho V_{\mathrm{B}}g$$
   and $$T = m_{\mathrm{A}}g - \rho f V_{\mathrm{A}}g$$ ✓

- So that $$m_{\mathrm{B}}g - \rho V_{\mathrm{B}}g = m_{\mathrm{A}}g - \rho f V_{\mathrm{A}}g$$ ✓

- Hence $$\rho f V_{\mathrm{A}}g = (m_{\mathrm{A}} - m_{\mathrm{B}})g + \rho V_{\mathrm{B}}g$$
   then $$f V_{\mathrm{A}} = \frac{(m_{\mathrm{A}} - m_{\mathrm{B}})}{\rho} + V_{\mathrm{B}}$$
   $$f V_{\mathrm{A}} = \frac{(500 - 390)}{1} + 50$$
   $$f1000 = 160$$
   and thus $$f = 0.16 = 16\%$$ ✓

- $$T = m_{\mathrm{B}}g - \rho V_{\mathrm{B}}g$$ ✓
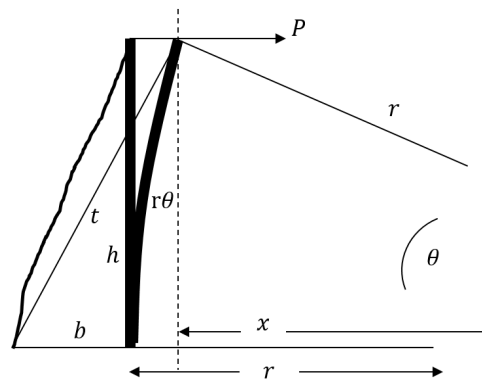   So $kx = (m_{\mathrm{B}} - \rho V_{\mathrm{B}})g$
   Now using units of m, kg, s, we have
   $$100x = (0.390 - 1000 \times 50 \times 10^{-6}) \times 9.8$$
   $$x = \frac{0.34 \times 9.8}{100} = 0.033\,\mathrm{m} = 3.3\,\mathrm{cm}$$ ✓

(6 marks)

---

t) Wooden pole.
   - Diagram to show idea and symbols ✓

- Diameter of pole $2a$.
- When pole is bent, inside arc length $= (r - a)\theta$
   and outside arc length $= (r + a)\theta$, so the extension will be $2\delta\ell$
   Subtracting $$2\delta\ell = 2a\theta$$ ✓
   Using $$\ell = h = r\theta \quad \text{in} \quad E = \text{stress.}\frac{\ell}{\delta\ell} = \text{stress.}\frac{r\theta}{a\theta}$$

- $r = \dfrac{aE}{\text{stress}} = \dfrac{0.15 \times 14 \times 10^9}{4000} = 5.25 \times 10^5\,\mathrm{m}$ ✓

- $\theta = \dfrac{\ell(= h)}{r} = \dfrac{12}{5.25 \times 10^5} = 2.29 \times 10^{-5}\,\mathrm{rad} = 1.31 \times 10^{-3}\,^\circ$ ✓

- $(r - x) = r(1 - \cos\theta) = r\dfrac{\theta^2}{2} = 1.38 \times 10^{-4}\,\mathrm{m}$

- slack length $= t - \sqrt{h^2 + b^2} = \sqrt{h^2 + (b + r - x)^2} - \sqrt{h^2 + b^2}$ ✓

There are two approaches to determining the slack length. Numerically and by algebraic approximations.

The mark for an order of magnitude result.

- numerical

  slack length $= \sqrt{12^2 + (2.4 + 1.38 \times 10^{-4})^2} - \sqrt{12^2 + 2.4^2}$

  $= 12.237674 - 12.237647$

  $= 2.7 \times 10^{-5}\,\text{m}$

- algebraic

$$= t - \sqrt{h^2 + b^2} = \sqrt{h^2 + (b + r - x)^2} - \sqrt{h^2 + b^2}$$

Using the binomial approximation:

$$= h\left(1 + \frac{1}{2}\left(\frac{b + r - x}{h}\right)^2 - 1 - \frac{1}{2}\left(\frac{b}{h}\right)^2\right)$$

$$= \frac{1}{2h}\left((b + r - x)^2 - b^2\right)$$

$$= \frac{1}{2h}\left(b^2 + (r - x)^2 + 2b(r - x) - b^2\right)$$

$$= \frac{(r - x)}{2h}(r - x + 2b)$$

$(r - x) \ll 2b$ and $r\dfrac{\theta^2}{2}$

$$= \frac{r\theta^2 2b}{2.2h} = \frac{rh^2 b}{2r^2 h} = \frac{bh}{2r}$$

$$= \tfrac{1}{2} 2.4 \times 2.29 \times 10^{-5}$$

$$= 2.7 \times 10^{-5}\,\text{m} \qquad\qquad \checkmark$$

(6 marks)

Qu 2

a) • Kinetic energy of wind transferred to turbine
$E_k = \frac{1}{2}mv^2$ ✓

• Take cylinder of air with a radius r and length l
Mass of air : $m = \rho V = \rho \pi r^2 l$

• Substitute into: $P = \frac{E}{t} = \frac{\rho \pi r^2 v^2}{2t}$ ✓

• Use $v = \dfrac{l}{t}$ to give: $P = \frac{1}{2}\rho \pi r^2 v^3$ ✓

(3 marks)

b) (i) $\rho A_1 v_1$

(ii) $\rho A v$

(iii) $\rho A_2 v_2$ ✓

(iv) • Conservation of mass flow $\rho A_1 v_1 = \rho A v = \rho A_2 v_2$
• Consider change in kinetic energy of the wind.
$\Delta E = \dfrac{1}{2}m(v_1^2 - v_2^2)$ ✓
• sub in $\dfrac{m}{\Delta t} = \rho A v$
$P_{\text{out}} = \dfrac{1}{2}\rho A v(v_1^2 - v_2^2)$ ✓

(v) Use $v = \dfrac{v_1 + v_2}{2}$

• $P_{out} = \dfrac{1}{4}\rho A(v_1 + v_2)(v_1^2 - v_2^2)$ ✓

(vi) • expand:
$P_{out} = \frac{1}{4}\rho A\left[v_1^3 - v_1 v_2^2 + v_2 v_1^2 - v_2^3\right]$
$P_{out} = \frac{1}{4}\rho A v_1^3\left[1 - (\frac{v_2}{v_1})^2 + (\frac{v_2}{v_1}) - (\frac{v_2}{v_1})^3\right]$ ✓
• Sub in $x = \dfrac{v_2}{v_1}$

$P_{out} = \frac{1}{4}\rho A v_1^3\left[1 - x^2 + x - x^3\right]$
• Differentiate and set equal to zero
$\dfrac{dP}{dx} = \frac{1}{4}\rho A v_1^3\left(-2x + 1 - 3x^2\right) = 0$
• Solve for x
$x = -1, \frac{1}{3}$ Take $\dfrac{v_2}{v_1} = \dfrac{1}{3}$ ✓

(vii) Use $\dfrac{v_2}{v_1} = \frac{1}{3}$
$P_{max} = \frac{1}{4}\rho.A.v_1^3\left[1 - (\frac{1}{3})^2 + (\frac{1}{3}) - (\frac{1}{3})^3\right]$
$P_{max} = 0.296\rho A v_1^3$ ✓

(viii) $P_{max} = 0.296 * 1.23 * (\pi * 107^2) * 10^3$
$P_{max} = 1.3 \times 10^7 \text{ W}$ ✓

ρ_{air} **was not given, so any value from 1 to 1.3 could be used here.**
If v_2 is taken out of the bracket then $P_{max} = 0.296\rho A v_2^3 = 35 \times 10^7$ W, which is $3^3 = 27$ times too large.
They only lose 1 mark for this as they have the method.

(8 marks)

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c) (i) • Resistance: $R = \dfrac{\rho L}{A} = \dfrac{1.68 \times 10^{-8} \times 265 \times 10^3}{\pi \times 0.022^2} = 2.93 = 2.9\,\Omega$ ✓

• Power Dissipated $P = I^2 R$

$I = \dfrac{P}{V} = \dfrac{800 \times 10^6}{320 \times 10^3} = 2500\,\text{A}$

$P = 2500^2\, 2.93 = 1.83 \times 10^7 W$ ✓

• Percentage Loss: % loss $= \dfrac{P_{heat}}{P_{carried}} = \dfrac{1.83 \times 10^7}{8 \times 10^8} \times 100 = 2.29 = 2.3\%$ ✓

(ii) • Resistance: $dR = \dfrac{\rho dr}{A} = \dfrac{\rho dr}{2\pi r L}$

Integrate

$$R = \frac{\rho}{2\pi L} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\rho}{2\pi L} \ln \frac{r_2}{r_1}$$

$$= \frac{1.97 \times 10^{14}}{2\pi 2.65 \times 10^5} \ln \frac{4}{2.2} = 7.07 \times 10^7 = 7.1 \times 10^7\,\Omega \quad ✓$$

OR A similar numerical answer can be obtained by taking the difference in the two radii $(1.8\,\text{cm})$ and using the average radius of $3.1\,\text{cm}$

Then $\qquad R = \dfrac{1.97 \times 10^{14} \times 1.8 \times 10^{-2}}{2\pi \times 3.1 \times 10^{-2} \times 2.65 \times 10^5} = 6.9 \times 10^7\,\Omega$

which would gave a current (below) of $4.6\,\text{mA}$, a power loss of $1500\,\text{W}$ and percentage loss of 2%.

Continuing

$I_{\text{leakage}} = \dfrac{V}{R} = \dfrac{320 \times 10^3}{7.1 \times 10^7} = 4.52 \times 10^{-3} = 4.5\,\text{mA}$ ✓

Loss $= I^2 R = (4.52 \times 10^{-3})^2 \times 7.07 \times 10^7 = 1400\,\text{W}$

Fractional loss $= \dfrac{1400}{800 \times 10^6} = 1.8 \times 10^{-6} = 2 \times 10^{-4}\%$ ✓

(iii) • We expect a small temperature difference so we can do an approximate calculation for the thermal conduction though a cylindrical surface.

Heat conductivity $P_{\text{heat loss}} = \dfrac{kA\Delta T}{\Delta r}$ A = area heat transferred $= 2\pi r_{\text{average}} L$

k = thermal conductivity

Δr = radial thickness of XPLE

So $1.83 \times 10^7 = 0.28 \times 2\pi \times 3.1 \times 10^{-2} \times 2.65 \times 10^5 \times \dfrac{(T_{\text{hot}} - T_{\text{cold}})}{1.8 \times 10^{-2}}$ ✓

$T_{\text{hot}} - 7\,^\circ\text{C} = 22.8$

$T_{\text{hot}} = 30\,^\circ\text{C}$ ✓

(8 marks)

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d) (i) work function for mercury was not given in question - give mark for any mention of work function or ionisation potential etc.

(ii) • Substitute equation for V_g

$I = a + bV_g + cV_g^2$

$= a + b(A + B\cos\omega t) + c(A + B\cos\omega t)^2$

$= a + bA + bB\cos\omega t + c(A^2 + 2AB\cos\omega t + B^2 \cos^2\omega t)$ ✓

• Use double angle trig identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ to give a term with $\cos(2\omega t)$:

$I = a + bA + cA^2 + \frac{1}{2}cB^2$

$\qquad + (bB + 2cAB)\cos\omega t$

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$$+ \frac{cB^2}{2} \cos 2\omega t$$

The amplitude of the second harmonic component is $\dfrac{cB^2}{2}$ ✓

- mean value of I is the sum of the DC terms as the $\cos \omega t$ and $\cos 2\omega t$ terms average to zero over many cycles ✓

So this also contains the B^2 term $\dfrac{cB^2}{2}$

- When $B = 0$, $I = a + bA + cA^2 = I_0$ ✓

So $I = I_0 + \dfrac{cB^2}{2} + (bB + 2cAB)\cos \omega t + \dfrac{cB^2}{2} \cos \omega t$

$$I_{\max} = I_0 + \frac{cB^2}{2} + (bB + 2cAB) + \frac{cB^2}{2} = I_0 + A_1 + 2A_2 \qquad ✓$$

$$I_{\min} = I_0 + \frac{cB^{\cancel{2}}}{\cancel{2}} - (bB + 2cAB) - \frac{cB^{\cancel{2}}}{\cancel{2}} = I_0 - A_1$$

$$I_{\max} - I_0 = A_1 + 2A_2$$

$$I_0 - I_{\min} = A_1$$

Hence $\chi = 1 + \dfrac{2A_2}{A_1}$

So $\dfrac{A_2}{A_1} = \dfrac{1}{2}(\chi - 1)$ ✓

(6 marks)

Qu 3.

a)
- Force down = weight + weight of hot air
$$= 200g + \rho_{hot}Vg$$ ✓
- Upthrust = weight of cold air displaced
$$= \rho_{cold}Vg$$
- Balance forces: $200g + \rho_{hot}Vg = \rho_{cold}Vg$ ✓

$$V = \frac{200}{\rho_{cold} - \rho_{hot}} = \frac{200}{1.23 - 0.95} = 714\,\text{m}^3$$ ✓

(3 marks)

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b)
- As in previous question, $830g + \rho_{hot}Vg = \rho_{cold}Vg$

So $\rho_{hot} = \dfrac{\rho_{cold}V - 830}{V} = \dfrac{1.23 \times 2000 - 830}{2000} = 0.815\,\text{kg m}^{-3}$ ✓

$$m_{hot\ air} = \rho_{hot}V = 0.815 \times 2000 = 1630\,\text{kg}$$

- $n = \dfrac{m_{hot\ air}}{0.029} = \dfrac{1630}{0.029} = 5.6 \times 10^4\,\text{mol}$ ✓

- Assuming ideal gas: $T = \dfrac{PV}{nR}$

$T = \dfrac{1.01 \times 10^5 \times 2000}{5.6 \times 10^4 \times 8.314} = 430\,\text{K} = 160\,°\text{C}$ ✓

(3 marks)

Alternatively:

[An estimate can be made using the ideal gas equation, $P \propto n/V\,RT$ and hence for a constant pressure, $\rho T = \text{const}$. If we assume a cold temperature of $20\,°\text{C}$ then
$$\rho_H T_H = \rho_C T_C$$
$$1.23 \times 293 = 0.815 \times T_H \qquad \text{so} \qquad T_H = 442\,\text{K} = 169\,°\text{C}$$
One mark for the earlier calculated density of hot air and 1 mark for the temperature calculation ie. 2/3]

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c) (i)
- Assuming ideal gas $n = \dfrac{PV}{RT}$

- $m = nM = \dfrac{PV}{RT}M$

$m = \dfrac{1.01 \times 10^5 \times 380}{8.314 \times 288} = 16030 \times 0.002 = 32.1 = 32\,\text{kg}$ ✓

(ii)
- Radius $r^3 = \dfrac{3 \times 380}{4 \times \pi}$

$r = 4.49\,\text{m}$ ✓

- calculate thicknesses of silk and rubber
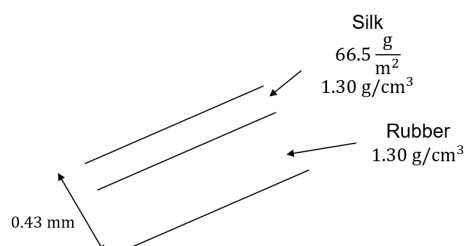
$$t_{silk} = \frac{66.5\,\text{g m}^{-2}}{1.30\,\text{g cm}^{-3}} = \frac{66.5 \times 10^{-5}\,\text{m}}{1.30} = 5.115 \times 10^{-5}\,\text{m}$$ ✓

So thickness of rubber $= 4.3 \times 10^{-4} - 0.5115 \times 10^{-4}\,\text{m} = 3.79 \times 10^{-4}\,\text{m}$ ✓

- Mass = thickness × density × area
  Mass of balloon $= (1.30 \times 10^3 \times 5.115 \times 10^{-5} + 1.30 \times 10^3 \times 3.788 \times 10^{-4})4\pi r^2$
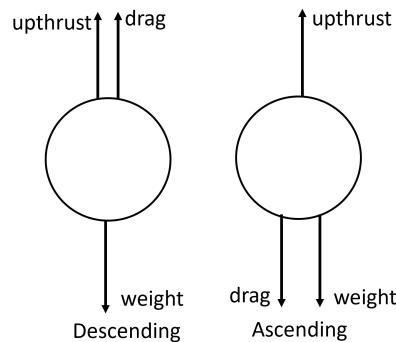 $$= 0.574 \times 4\pi 4.49^2 = 145.6 = 146 = 150\,\text{kg}$$ ✓

- $m_{\text{total}} = m_{\text{Hydrogen}} + m_{\text{balloon}} + m_{\text{payload}}$
  mass $= (32 + 146 + 270) = 448\,\text{kg}$
  So $\qquad 448a = 380 \times 1.23 \times 9.81 - 448 \times 9.81$
  $$a = 0.42\,\text{m s}^{-2}$$ ✓

(6 marks)

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d) Descends with constant velocity forces are balanced, drag always in opposite direction to travel.

- Since $F_d \propto v$ then the drag force is the same going down as up, since v is the same.
- Descending: $F_d + U = W$ ✓

- Ascending: $W - m_2 g + F_d = U$ ✓

- Re-arranging both for U as upthrust is constant can set equal.
  $W - m_2 g + F_d = W - F_d$
  $$m_2 = \frac{2F_d}{g}$$ ✓

(3 marks)

---

e) (i)   - $\Delta p = \rho g \Delta h$

- Ideal gas equation: $pV = nRT$ and M is the kg molar mass.
  Use $V = \dfrac{m}{\rho}$ and $n = \dfrac{m}{M}$ ✓

- $p\dfrac{m}{\rho} = \dfrac{m}{M}RT$
  $$p = \frac{\rho RT}{M}$$ ✓

- Substitute equation for ρ in ideal gas equation into hydrostatic equation:
  $$\frac{\Delta p}{p} = \frac{Mg}{RT}\Delta h$$ ✓

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- The differential form includes a $-$ sign since as z increases P decreases.

  Then $\qquad dp = -\rho g dz = -\dfrac{pM}{RT}g dh$

  Substitute $T = T_0 - \alpha h$ into this result:

  $\dfrac{\Delta p}{p} = -\dfrac{Mg}{R(T_0 - \alpha h)}\Delta h$

- Integrate both sides:

$$\int_{P_0}^{P(h)} \frac{1}{p}dp = \int_0^h -\frac{Mg}{R(T_0 - \alpha h)}dh \qquad \checkmark$$

$$\ln\left(\frac{P(h)}{P_0}\right) = \frac{Mg}{R\alpha}\ln\left(\frac{T_0 - \alpha h}{T_0}\right)$$

$$P(h) = P_0\left(\frac{T_0 - \alpha h}{T_0}\right)^{\frac{Mg}{R\alpha}} \qquad \checkmark$$

(5 marks)

(ii)
- Get temperature from $T = T_0 - \alpha h = 288 - (0.00976 \times 3000) = 258.7 = 259\,\text{K}$ $\checkmark$

- Substitute values into the equation derived:

$$P(3000) = 1.01 \times 10^5 \left(\frac{288 - 0.00976 \times 3000}{288}\right)^{\frac{0.0299 \times 9.81}{8.314 \times 0.00976}}$$

$$P(3000) = 1.01 \times 10^5 \left(\frac{259}{288}\right)^{3.51} = 7.0 \times 10^4\,\text{Pa} \qquad \checkmark$$

(2 marks)

(iii)
- Pressure inside balloon: $\dfrac{P_1 V_1}{T_1} = \dfrac{P_2 V_2}{T_2}$

  $\dfrac{1.01 \times 10^5 \times V_1}{288} = \dfrac{P_2 \times 1.3 \times V_1}{259}$ gives $P_2 = 70\,\text{kPa}$ $\qquad \checkmark$

  OR

  If the temperature of the hydrogen remains at $288\,\text{K}$ then the pressure is given by $1.01 \times 10^5 \times V_1 = P_2 \times 1.3 V_1$ and $P_2 = 78\,\text{kPa}$

- Stress is given by the (pressure) $\times$ (area of cross section) $\div$ (circumferential area of material)

$$= \frac{P_2 \times \pi r^2}{2\pi r \delta r} = \frac{P_2 r}{2\delta r} \qquad \checkmark$$

$$= \frac{0.70 \times 10^5 \times [4.49 \times \left(1.3\right)^{1/3}]}{2 \times 0.43 \times 10^{-3}} = 4 \times 10^8\,\text{Pa} \qquad \checkmark$$

- Young's Modulus $E = \dfrac{stress}{strain}$

- In the surface of the balloon, strain $= \dfrac{\ell'}{\ell_o} = \dfrac{r'}{r_o} = \left(\dfrac{V'}{V_o}\right)^{\frac{1}{3}} = 1.3^{\frac{1}{3}} = 1.09$

  $\dfrac{\delta \ell}{\ell_o} = \dfrac{\ell' - \ell_o}{\ell} = 1.09 - 1 = 0.09$

- so $E = \dfrac{4 \times 10^8}{0.09} = 4 \times 10^9\,\text{Pa}$ $\qquad \checkmark$

(4 marks)

Qu 4.

a)   (i)   • Gravitational field strength: $g_{\mathrm{E}} = \dfrac{GM}{R_{\mathrm{E}}^2}$

        • When $g = 0.01 g_E$ then $R \rightarrow 10 R_{\mathrm{E}}$ ✓

  (ii)   • Angular Speed of earth : $\omega = \dfrac{2\pi}{24 \times 3600}$ ✓

        • Tangential Speed, $v = \omega R_{\mathrm{E}} \cos\phi$

$$v = \frac{2\pi}{24 \times 3600} \times 6.37 \times 10^6 \times \cos 30 = 400 \,\mathrm{m\,s^{-1}}$$ ✓

  (iii)   • Escape velocity is reached when kinetic energy is equal to potential energy.

$$\tfrac{1}{2}mv_{es}^2 = \frac{GMm}{R_{\mathrm{E}}}$$ ✓

$$v = \sqrt{\frac{2GM}{R_{\mathrm{E}}}} = \sqrt{2 g_{\mathrm{E}} R_{\mathrm{E}}} = 11.2 \,\mathrm{km\,s^{-1}}$$

$$= \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.37 \times 10^6}} = 11.2 \,\mathrm{km\,s^{-1}}$$ ✓

  (iv)   i. $\tfrac{1}{2}mv_{\mathrm{new.es}}^2 + \tfrac{1}{2}mv_{Taneg}^2 = \dfrac{GMm}{R_{\mathrm{E}}} = \tfrac{1}{2}mv_{es}^2$

        So $v_{\mathrm{new.es}}^2 + v_{Taneg}^2 = v_{es}^2$ ✓

        • Reduction in energy "TO" $= \dfrac{v_{\mathrm{new.es}}^2}{v_{es}^2} = \dfrac{1 - v_{Taneg}^2}{v_{es}^2}$

$$= 1 - \frac{0.4^2}{11.2^2} \times 100 = \textcolor{red}{98.7\,\%} \text{ or "BY" } \textcolor{red}{0.013\,\%}$$ ✓

        ii. In low Earth orbit,

$$\frac{v_{\mathrm{lowE}}^2}{R_{\mathrm{E}}} = \frac{GM_{\mathrm{E}}}{R_{\mathrm{E}}^2} \quad \text{so that} \quad v_{\mathrm{lowE}} = \sqrt{\frac{GM_{\mathrm{E}}}{R_{\mathrm{E}}}}$$

        Hence the energy is reduced by $\dfrac{v_{es}^2 - v_{\mathrm{lowE}}^2}{v_{es}^2} = 1 - \dfrac{\frac{GM_{\mathrm{E}}}{R_{\mathrm{E}}}}{\frac{2GM_{\mathrm{E}}}{R_{\mathrm{E}}}}$

$$= 1 - \tfrac{1}{2} = 50\%$$ ✓

<div align="right">(8 marks)</div>

  (v)   • Conservation of Energy: $(KE)_P + (PE)_P = (KE)_A + (PE)_A$

$$\tfrac{1}{2}mv_P^2 - \frac{GM_s m}{R_{\mathrm{E_o}}} = \tfrac{1}{2}mv_A^2 - \frac{GM_s m}{R_{\mathrm{M_o}}}$$ ✓

        • Rearrange for v_P^2

$$v_P^2 = v_A^2 + 2GM_s \left(\frac{1}{R_{\mathrm{M_o}}} - \frac{1}{R_{\mathrm{E_o}}} \right)$$

        • Conservation of angular momentum at these tangential speeds at the ends of the ellipse
        where $\sin\theta = 1$

        momentum: $R_{\mathrm{E_o}} v_P = R_{\mathrm{M_o}} v_A$

        Hence $v_P^2 = v_P^2 \dfrac{R_{\mathrm{E_o}}^2}{R_{\mathrm{M_o}}^2} + 2GM_s (R_{\mathrm{M_o}} - R_{\mathrm{E_o}})$

$$\frac{v_P^2}{R_{\mathrm{M_o}}^2}(R_{\mathrm{M_o}} - R_{\mathrm{E_o}})(R_{\mathrm{M_o}} + R_{\mathrm{E_o}}) = 2GM_s \frac{(R_{\mathrm{M_o}} - R_{\mathrm{E_o}})}{R_{\mathrm{M_o}} R_{\mathrm{E_o}}}$$ ✓

$$v_P = \sqrt{\frac{2GM_s}{(R_{\mathrm{M_o}} + R_{\mathrm{E_o}})} \frac{R_{\mathrm{M_o}}}{R_{\mathrm{E_o}}}}$$ ✓

<div align="center">20</div>

and
$$v_A = \sqrt{\frac{2GM_s}{(R_{M_o} + R_{E_o})}\frac{R_{E_o}}{R_{M_o}}}$$

Earth's circular orbital speed is given by

$$\frac{v_E^2}{R_{E_o}} = \frac{GM_s}{R_E^2} \quad \rightarrow \quad v_{E_o} = \sqrt{\frac{GM_s}{R_{E_o}}} \qquad \checkmark$$

and Mars's orbital speed is
$$v_M = \sqrt{\frac{GM_s}{R_{M_o}}} \qquad \text{(one mark for either)}$$

(vi) & (vii) The question is unclear as to what is being looked for (the speed to catch Mars or the total increment in speed). The probe has to be accelerated to catch Mars when it reaches the same orbital radius. There was a mistake in the algebra earlier for the differences in speeds, but the equations for v_P, v_E, v_A, v_M were all correct. So here are the calculated values.
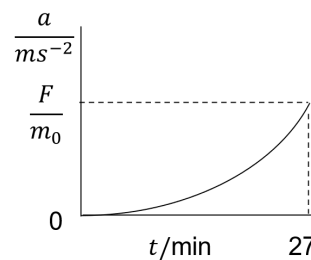
$$v_A = 21.6 = 22\,\text{km}\,\text{s}^{-1} \qquad \checkmark$$
$$v_M = 24.2 = 24\,\text{km}\,\text{s}^{-1}$$
$$v_P = 32.6 = 33\,\text{km}\,\text{s}^{-1} \qquad \checkmark$$
$$v_E = 29.7 = 30\,\text{km}\,\text{s}^{-1}$$

- In the section below, there is a mark for the graph as before, and a mark for the method to calculate the thrust.

(viii)   • Diagram $\qquad \checkmark$

   •
$$dv = a\,dt = \frac{F}{m}\,dt$$

$m = m_0 - kt$ with $k = \dfrac{\delta m}{t}$

F is in same direction as v

$$\int_v^{v_0} dv = \int_0^t \frac{F}{m_0 - kt}\,dt$$

$$v_0 - v = -\frac{F}{k}\ln\left(1 - \frac{kt}{m_0}\right)$$

$$\Delta v = -\frac{Ft}{\delta m}\ln\left(1 - \frac{\delta m}{m_0}\right)$$

Mark for method above $\quad \checkmark$

The $\quad \Delta v = v_M - v_A = 24.2 - 21.6 = 2.6\,\text{km}\,\text{s}^{-1}$

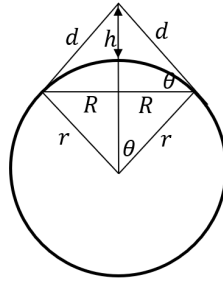$$2.6 \times 10^3 = -\frac{F \times 27 \times 60}{400}\ln\left(1 - \frac{400}{1350}\right)$$

$F = 1.8\,\text{kN}$ $\qquad$ mark for method not the numeric result

So for one thruster the force is $\approx 0.5\,\text{kN}$

(8 marks)

b)

(i) Diagram ✓

$$\cos\theta = \frac{r}{h+r}$$

$$\sin\theta = \frac{R}{r}$$

Using the identity $s^2 + c^2 = 1$, we have

$$\frac{r^2}{(h+r)^2} + \frac{R^2}{r^2} = 1 \qquad R, r \text{ and } h \text{ related}$$ ✓

So $r^2 r^2 + R^2(h+r)^2 = r^2(h+r)^2$

expanding $\quad r^4 + R^2(h^2 + r^2 + 2hr) = r^2 h^2 + r^4 + 2hr^3$

Hence $\quad R^2 = r^2 \dfrac{(h^2 + 2hr)}{(h^2 + 2hr + r^2)}$

$$= 6370^2 \frac{(780^2 + 2 \times 780 \times 5370)}{(780^2 + 2 \times 780 \times 5370 + 6370^2)}$$

$R = 2890\,\text{km}$ ✓

(ii) For $R = 2350\,\text{km}$ the intensity is

$$I \approx \frac{200}{\pi R^2} = 1.2 \times 10^{-11}\,\text{W m}^{-2}$$ ✓

(iii) Some realistic derivation of the form,

in a time Δt there are $f_\text{s}\Delta t$ wavelengths emitted from a receding source which are received by a stationary observer.

These wavelengths are covering a distance $(c+v)\Delta t$

The length of one of these wavelengths, $\lambda_\text{o} = \dfrac{(c+v)\Delta t}{f_\text{s}\Delta t}$

Thus $\quad \lambda_\text{o} = \dfrac{(c+v)}{f_\text{s}} = \lambda_\text{s} + v/f_\text{s} = \lambda_\text{s} + v\lambda_\text{s}/c = \lambda_\text{s}(1 + v/c)$ ✓

(iv) $\dfrac{\Delta f}{f} = \dfrac{v}{c}$ for $v \ll c$. ✓

(v) As the satellite passes overhead, the observer will not see it approaching or receding and will conclude that there is no Doppler effect. As the satellite rises above the horizon on one side or goes below the horizon on the other. it will be approaching or receding at the maximum speed from the observer on the Earth's surface.

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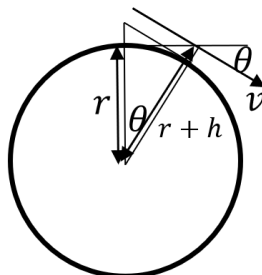
$v_{\text{recession}} = v \cos \theta$

$\cos \theta = \dfrac{r}{(r + h)} = \dfrac{6370}{(6370 + 780)} = \dfrac{6370}{7150}$

$\cos \theta = 0.89$ ✓

Hence $v_{\text{recession}} = v \times 0.89$

For the orbital speed v, using $\dfrac{v^2}{r} = \dfrac{GM}{r^2}$ then $v^2 = \dfrac{GM}{r} = \dfrac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{7.15 \times 10^6}$

This gives $v = 7460 \, \text{m s}^{-1}$ ✓

The resulting maximum velocity of recession or approach is

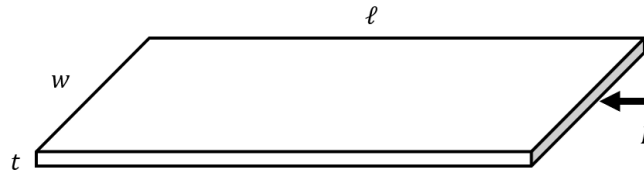$v_{\text{recession}} = 7460 \times 0.89 = 6640 \, \text{m s}^{-1}$

The Doppler frequency shift will be $\Delta f = \dfrac{fv}{c} = \dfrac{1620 \times 10^6 \times 6640}{3 \times 10^8} = \pm 36 \, \text{kHz}$ ✓

(9 marks)

Qu 5.

a)

(i) With
$$R = \frac{\rho \ell}{wt} = \frac{1.5 \times 10^{-7} \times 0.3}{4 \times 10^{-3} \times 10 \times 10^{-6}}$$
$$= 1.125 \, \Omega$$
$$= 1.13 = 1.1 \, \Omega \qquad ✓$$

(ii) Substituting into $R = R_0(1 + \alpha\theta)$
$$R = 1.125(1 + 3.3 \times 10^{-3} \times 15)$$
$$R = 1.18 \, \Omega = 1.2 \, \Omega \qquad ✓$$

Hence $P = \dfrac{V^2}{R}$ gives $P = 3.39 = 3.4 \, \text{W} \qquad ✓$

(iii) Power loss $= kA(\theta - 0) = kA\theta \qquad ✓$

So
$$\frac{V^2}{R_0(1 + \alpha\theta)} = kA\theta$$

This gives $kAR_0\alpha\theta^2 + R_0 kA\theta - V^2 = 0 \qquad ✓$

$$\theta = \frac{-R_0 kA \pm \sqrt{R_0^2 k^2 A^2 + 4R_0\alpha kAV^2}}{2R_0\alpha kA}$$

$$= -\frac{1}{2\alpha} \pm \frac{R_0 kA}{2\alpha R_0 kA}\sqrt{1 + \frac{4\alpha V^2}{R_0 kA}}$$

$$= \frac{1}{2\alpha}\left(-1 + \sqrt{1 + \frac{4\alpha V^2}{R_0 kA}}\right)$$

$$= \frac{1}{2 \times 3.3 \times 10^{-3}}\left(-1 + \sqrt{1 + \frac{4 \times 3.3 \times 10^{-3} \times 4}{1.125 \times 22 \times 0.3 \times 4 \times 10^{-3} \times 2}}\right)$$

$= 57 \, °\text{C}$ if front and back surfaces considered ($2A$)

or $101 \, °\text{C}$ if only one surface is considered for A. Either answer is allowed. $\qquad ✓$

(6 marks)

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b) The filament has a transverse wave with a node in the middle and four loops
So that means that $2\lambda = 25 \, \text{cm}$ $\qquad ✓$

The speed of a transverse wave along a wire under tension T is $v = \sqrt{\dfrac{T}{\mu}}$

So that using $v = f\lambda$ we can write $f^2\lambda^2 = \dfrac{T}{\mu}$

We will need to know the mass per unit length, μ, and we are given the density. We need to estimate the diameter of the filament.

It is greater than $0.1 \, \text{mm}$ and less then $10 \, \text{mm}$ so we can take $1 \, \text{mm}$. $\qquad ✓$

$$\mu = \frac{m}{\ell} = \frac{\rho A\ell}{\ell} = \rho A = \rho\pi\frac{\ell^2}{2}$$

An estimate of μ is $19 \times 10^3 \times \pi \times (0.5 \times 10^{-3})^2 = 0.015 \, \text{kg m}^{-2}$

Hence $T = (50 \times \dfrac{0.25}{2})^2 \times 0.015 = 0.6 \, \text{N}$ $\qquad ✓$

(3 marks)

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c) $J = \dfrac{I}{A}$ and $P = I^2 R$

So $P = I^2 R = J^2 A^2 \times \dfrac{\rho \ell}{A} = J^2 A \rho \ell = J^2 V \rho$ ✓

(1 marks)

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d) For a balanced circuit $\dfrac{R_2}{R_{\text{bulb}}} = \dfrac{R_1}{R_3}$. In this case that means the the bulb must be the same as R_2,

i.e. $4\,\Omega$ ✓

So $R_{\text{bulb}} = 4\,\Omega = \dfrac{V}{I} = \dfrac{2I + 8I^2}{I} = 2 + 8I$ ✓

this gives $I_{\text{bulb}} = 0.25\,\text{A}$

Each resistor has $0.25\,\text{A}$ flowing through it, so $V_{\text{b}} = 2\,\text{V}$ ✓

(3 marks)

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e) Given that the maximum power converted in the cell is when the external resistance is equal to the internal resistance R,

(notation "//" reads "is in parallel with"

(i) We have $\qquad (x+1)R // yR + xR = R$ ✓

Hence $\qquad \dfrac{(x+1)R.yR}{(x+1)R + yR} + xR = R$

Cancelling $R's$ we have

$\dfrac{(x+1)y}{(x+y+1)} + x = 1$

then $\qquad xy + y + x(x+y+1) = x + y + 1$

which gives $\qquad xy + \cancel{y} + x^2 + \cancel{x} + xy = \cancel{x} + \cancel{y} + 1$

$x(2y + x) = 1$

$2xy + x^2 - 1 = 0$ ✓

(ii) Kirchhoff I for currents at a node $i_1 = i_2 + i_3$

then $\qquad \dfrac{i_1}{i_2} = 1 + \dfrac{i_3}{i_2}$ ✓

So then $\qquad \dfrac{i_1}{i_2} = 1 + \dfrac{i_3}{i_2} = 1 + \dfrac{V}{yR} \Big/ \dfrac{V}{xR + R}$

$\qquad\qquad\qquad = 1 + \dfrac{x+1}{y} = \dfrac{(x+y+1)}{y}$ ✓

(iii) • P_{input} is power delivered by the cell $(= Ei_1 - i_1^2 R)$

but this is equal to the power dissipated in all the resistors. ✓

• Hence the required quantity is $\dfrac{i_1^2 xR + i_2^2(x+1)R + i_3^2 yR}{i_2^2 R}$ ✓

$= \dfrac{i_1^2}{i_2^2} x + (x+1) + \dfrac{i_3^2}{i_2^2}$

$= \dfrac{x}{y^2}(y + x + 1)^2 + (x+1) + \dfrac{y}{y^2}(x+1)^2$

using the previous results for $\dfrac{i_1}{i_2}$ and $\dfrac{i_3}{i_2} = \dfrac{(y+1)}{y}$

Now it is algebra and using the relation between x and y in part (i)

$\dfrac{P_{\text{input}}}{P_{\text{load}}} = \dfrac{x}{y^2}\left(\dfrac{1 - x^2}{2x} + x + 1\right)^2 + x + 1 + \dfrac{2x}{1 - x^2}(x+1)^2$

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$$= \frac{x}{y^2}\left(\frac{1-x^2+2x^2+2x}{2x}\right)^2 + x + 1 + 2x\frac{(1+x)}{(1-x)}$$

$$= \frac{x}{y^2}\left(\frac{x^2+2x+1}{2x}\right)^2 + \frac{(1-x^2+2x^2+2x)}{(1-x)}$$

$$= \frac{x}{y^2}\frac{(x+1)^4}{4x^2} + \frac{(1+x)^2}{(1-x)}$$

$$= (x+1)^2\left(\frac{(x+1)^2}{4xy^2} + \frac{1}{(1-x)}\right)$$

$$= (x+1)^2\left(\frac{(x+1)^2 4x^2}{4x(1-x^2)^2} + \frac{1}{(1-x)}\right)$$

$$= \frac{(x+1)^2}{(1-x)}\left(\frac{\cancel{(x+1)^2}x}{(1-x)\cancel{(1+x)^2}} + 1\right)$$

$$= \frac{(x+1)^2}{(1-x)}\left(\frac{x+(1-x)}{(1-x)}\right)$$

$$= \frac{(1+x)^2}{(1-x)^2} \qquad \checkmark$$

(7 marks)

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f) (i) The resistance between the two hemispheres is $R = \dfrac{\rho\delta r}{2\pi r^2}$ $\qquad\checkmark$

(ii) For a large value of δr the area will change and so we need to sum the resistance contributions from each hemispherical layer of liquid.

Integrating up $\quad dR = \dfrac{\rho dr}{2\pi r^2}$ for a sphere of radius r_s to r_large

$$\int_{R_\text{s}}^{R_\text{large}} dR = \int_{r_\text{s}}^{r_\text{large}} \frac{\rho dr}{2\pi r^2}$$

$\qquad\checkmark$

$$R_\text{large} - R_\text{s} = \Delta R = -\frac{\rho}{2\pi}\left[\frac{1}{r}\right]_{r_\text{s}}^{r_\text{large}}$$

$$\Delta R = \frac{\rho}{2\pi}\left(\frac{1}{r_\text{s}} - \frac{1}{r_\text{large}}\right)$$

As r_large tends to infinity then $1/r_\text{large}$ tends to zero. The resistance between the sphere and a distant surface (the tank) becomes $\Delta R = \dfrac{\rho}{2\pi r_\text{s}} = 95 = 100\,\Omega$, $\qquad\checkmark$

(iii) If a sphere is used instead, then the current flow lines will be the same but there will be two identical paths for the current flowing radially outwards from the sphere. So the resistance will be halved to $\Delta R = \dfrac{\rho}{4\pi r_\text{s}} = 50\,\Omega$ $\qquad\checkmark$

(iv) $\dfrac{\rho}{2\pi r}$ $\qquad\checkmark$

(5 marks)

END OF SOLUTIONS