British Physics Olympiad 2017-18

Round 2 Competition Paper

Monday 29th January 2018

Instructions

Time: 3 hours (approximately 35 minutes per question).

Questions: All five questions should be attempted.

Marks: The questions carry similar marks.

Solutions: Answers and calculations are to be written on loose paper or in examination booklets, and graph paper should be provided. Students should ensure their name and school is clearly written on all answer sheets and pages are numbered. A standard formula booklet with standard physical constants should be supplied.

Instructions: To accommodate students sitting the paper at different times, please do not discuss any aspect of the paper on the internet until 8 am Saturday 3rd February.

This paper must not be taken out of the exam room.

Clarity: Solutions must be written legibly, in black pen (the papers are photocopied), and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam paper.

Training Dates and the International Physics Olympiad

Following this round, the best students eligible to represent the UK at the International Physics Olympiad (IPhO) will be invited to attend the **Training Camp** *to be held in the Physics Department at the University of Oxford,* **(Monday 9th April to Friday 13th April 2018)**. *Problem solving skills will be developed, practical skills enhanced, as well as some coverage of new material (Thermodynamics, Relativity, etc.). At the Training Camp a practical exam is sat as well as a short Theory Paper. Five students (and a reserve) will be selected for further training. From May there will be mentoring by email to cover some topics and problems. There will be a weekend* **Experimental Training Camp in Oxford 18th – 20th May (Friday evening to Sunday afternoon)**, followed by a **Training Camp in Cambridge beginning on Thursday 28th June.**

The IPhO this year will be held in Lisbon, Portugal, from 21th to 29th July 2018.

Important Constants

Constant	**Symbol**	**Value**
Speed of light in free space	c	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Elementary charge	e	$1.60 \times 10^{-19}\,\mathrm{C}$
Acceleration of free fall at Earth's surface	g	$9.81\,\mathrm{m\,s^{-2}}$
Permittivity of free space	ε_0	$8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	$4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Mass of an electron	m_{e}	$9.11 \times 10^{-31}\,\mathrm{kg}$
Mass of a neutron	m_{n}	$1.67 \times 10^{-27}\,\mathrm{kg}$
Mass of a proton	m_{p}	$1.67 \times 10^{-27}\,\mathrm{kg}$
Boltzmann constant	k	$1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Planck constant	h	$6.63 \times 10^{-34}\,\mathrm{J\,s}$
Gravitational constant	G	$6.67 \times 10^{-11}\,\mathrm{m^{3}\,kg^{-1}\,s^{-2}}$

BPhO Sponsors

Qu 1. General questions

(a). In 1927 a famous conference on Photons and Electrons was held in Bruxelles, the 5th Solvay Conference. The leading physicists of the day, in the rapidly developing field of Quantum Mechanics, attended. A photo of the conference delegates is shown in **Figure 1** below. Write down as many names of the physicists from that era who you think might be in the photo. You may include relevant names, even if you may not be able to recognise them in the photo.

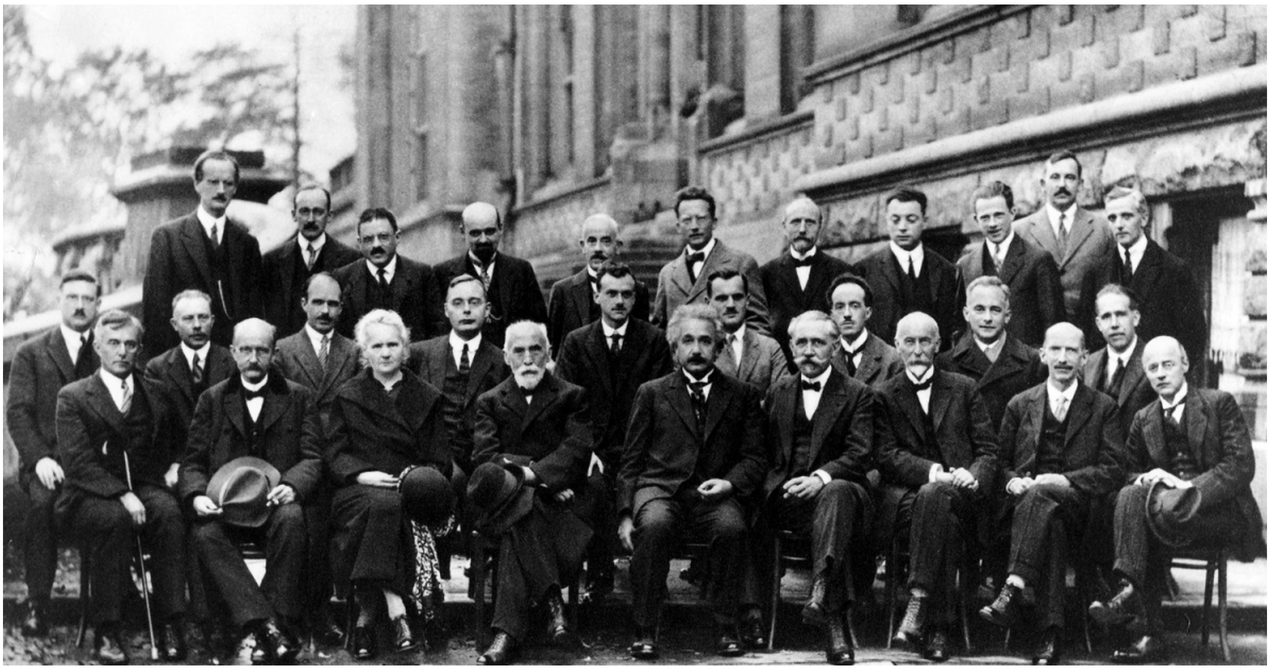


Figure 1: Physicists who took part in the 5th Solvay Conference in Bruxelles 1927

(b). In **Figs 2a** and **2b** are two pictures taken of the same lens resting on a sheet of newsprint. The camera used was held many focal lengths above the lens. What can you deduce about the lens from these observations. No calculations are required, but diagrams may be helpful.

(a) **(b)**

Figure 2: Photos of each side of a single lens resting on newsprint.

(c). Estimate the mass of a patch of mist which forms on a cold window when you breathe on it.

(d). An interesting paradoxical two terminal circuit formed from passive components can be seen in the following example. Two $1.0\,\Omega$ resistors and two $1.0\,\mathrm{V}$ zener diodes, Z_1 and Z_2, are connected in a manner known as a Wheatstone bridge, in **Figure 3a**. (A typical diode will not conduct in the reverse direction, but a zener diode will conduct in the reverse direction if the potential difference across it is greater than a specified voltage; e.g. an ideal 1.4 V zener diode will conduct perfectly in the forward direction, and will also conduct perfectly in reverse if the reverse the potential across it is greater than 1.4 V.)

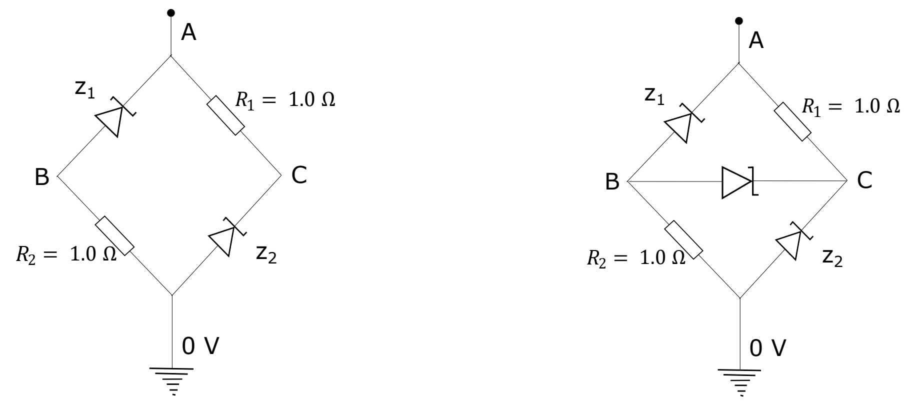
(i) In **Figure 3a**, a +2 V potential from a power supply is connected to point A. What current flows through each branch of the circuit down to earth?

(ii) In **Figure 3b**, a $\frac{3}{8}$ V zener diode, Z_3, is connected as shown. The same +2 V potential from a power supply connected to point A. What current now flows through the left and right hand arms of the circuit (i.e. paths AB0 and AC0)?

(iii) In **Figure 3a**, a power supply connected to the arrangement shown. A current of $\frac{1}{2}$ A flows into the circuit at A and flows down to earth. What are the potentials at points A, B and C?

(iv) In **Figure 3b**, the $\frac{3}{8}$ V zener diode Z_3 is connected as shown. The current from the supply is maintained at $\frac{1}{2}$ A. What are the potentials at points A, B and C now?

(v) Comment on what you might see as the paradox in this circuit.



(a) Wheatstone bridge arrangement of two resistors and two zener diodes

(b) Bridge with a third zener.

Figure 3: Ref. Paradoxical Behaviour of Mechanical and Electrical Networks, *Cohen JE, Horowitz P*, Nature v352 Aug 1992 p699.

(e). A small bead of mass m can slide along a loop of wire bent in the form of two 90° curves joined by a straight section, shown in **Fig. 4**. Gravity does not apply here. As the bead slides, it is only slowed down by a contact frictional force given by $F = \mu N$, where N is the normal force as it goes round the curve. If the bead has an initial speed $v_0 = 1.4\,\mathrm{m\,s^{-1}}$ and $\mu = 0.25$, what would be its speed v after it exits the second bend in the wire? The radius of the bead is much smaller than the radius of a bend.

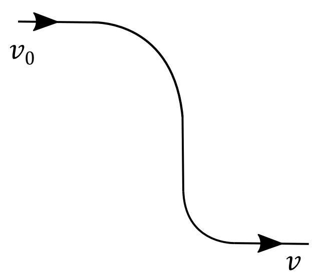


Figure 4

Qu 2. Gravitational Waves

Gravitational waves were first detected on the 14th September 2015 by the LIGO collaboration using two sets of large detectors based on the interference of waves. In this question we investigate some of the physics involved, using a combination of approximating the relativistic physics classically and dimensional analysis. The observed event, called GW150914, occurred as a result of a binary black hole pair merging to form a single black hole whilst simultaneously releasing a lot of energy in the form of gravitational waves.

(a). Before the two black holes merge, they can be modelled as two point masses M and m that rotate about their common centre of mass with radii R and r, respectively. Assume that the binary black holes obey Newtonian mechanics and gravity. Show that the angular frequency ω of the motion satisfies

$$\omega^2 = \frac{G(M+m)}{(R+r)^3},$$

where G is Newton's universal gravitational constant.

(b). Find the total energy of the black hole system and express this in terms of the masses of the black holes and the angular frequency of the motion.

Unlike Newtonian gravity, Einstein's general relativity predicts that accelerating objects will radiate gravitational waves and thus the binary black hole system will lose energy and the black holes will spiral into each other. The rate of radiation depends on the quadruple moment of the system. For our purposes, the quadruple moment can be considered as the same as the moment of inertia I of the system about its centre of mass. The radiated power P is proportional to I^2 and also depends on the angular frequency ω, and on the fundamental constants G and c, the speed of light. We shall use dimensional analysis to determine the rate of loss in terms of these quantities.

(c). Find the moment of inertia I of the black hole system about its centre of mass.

(d). The radiated power loss P can be written as

$$P = \alpha I^2 G^\beta \omega^\gamma c^\delta.$$

Find the exponents β, γ and δ. More detailed analysis shows that $\alpha = 32/5$.

(e). Find the rate of change in the angular frequency $\frac{\mathrm{d}\omega}{\mathrm{d}t}$ in terms of constants, ω and the *chirp mass*

$$\mathcal{M} = \frac{(mM)^{3/5}}{(m+M)^{1/5}}.$$

The quadruple moment is symmetric under rotations by π and so the frequency of the gravitational waves f is *twice* that of the orbital frequency.

(f). By integrating your answer to (e), show that over any interval of time τ the observed frequencies, f_1 and f_2 satisfy

$$\frac{1}{f_1^{8/3}} - \frac{1}{f_2^{8/3}} = 8\alpha\pi^{8/3}\frac{(G\mathcal{M})^{5/3}}{c^5}\tau.$$

(g). Using the data from LIGO, given in **Figure 5**, estimate the chirp mass $\mathcal{M}$, expressing your answer in terms of Solar masses $M_\odot = 1.989 \times 10^{30}$ kg.

(h). Deduce the lower bound on the total mass $m + M$ of the black holes before coalescence using your estimate of $\mathcal{M}$.

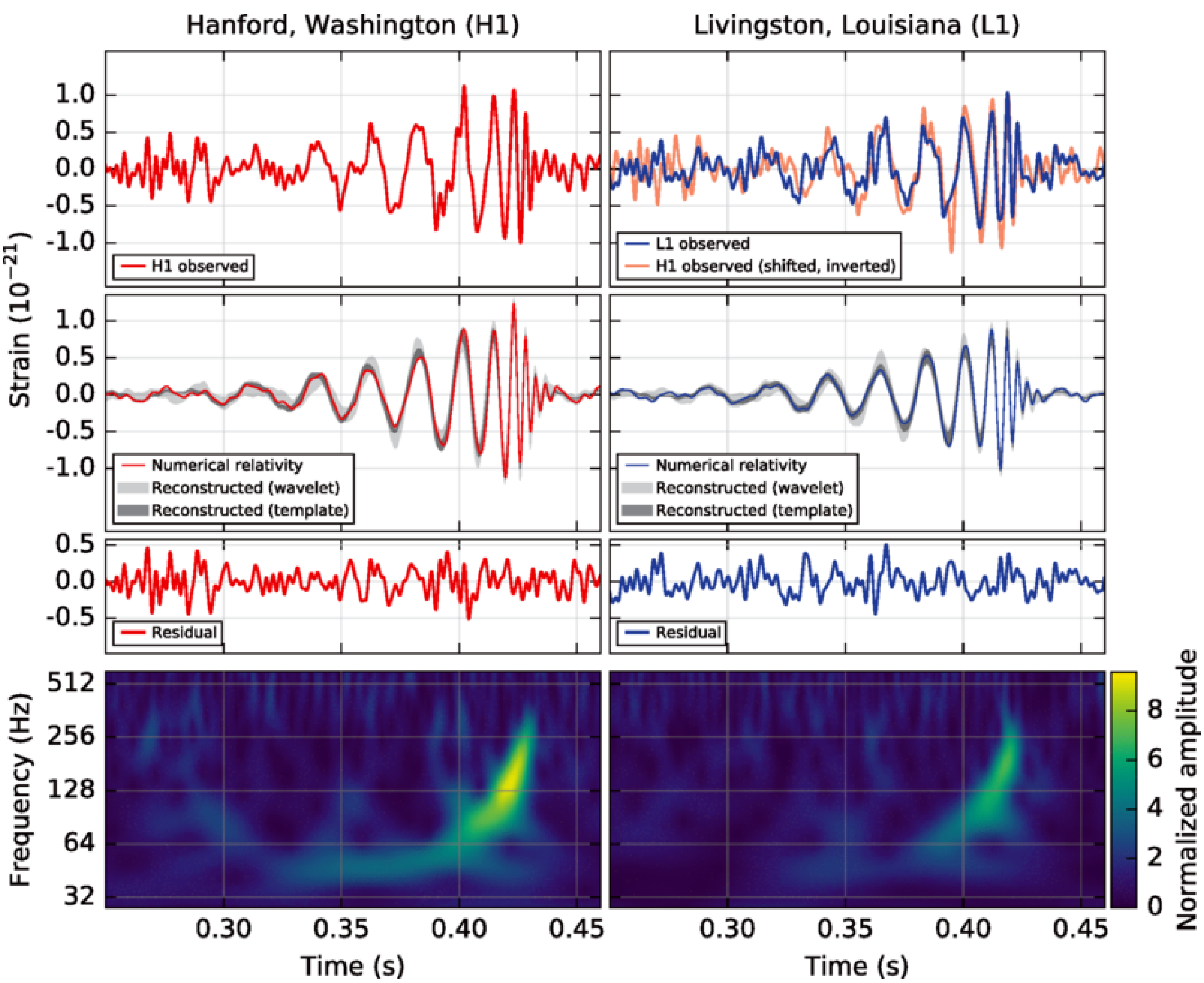


Figure 5: Data from Abbott *et al.*, Observation of gravitational waves from a binary black hole merger, PRL **116**, 061102 (2016). The left hand side panels refer to measurements made by the Hanford detector and the right hand side from Livingston. The top two rows show measurements and simulations of the strain on the detectors as a function of time. The third row details the differences between the simulation and measurements. The fourth row provides the relative amplitudes of the strain signal of the component frequencies as a function of time. In particular for part (g), note the signal frequency increasing in time.

In order to obtain better estimates of the total mass as well as the individual masses, we need to know something about the sizes of black holes. The radius of a black hole, where light itself cannot escape, is called the *Schwarzschild* radius R_S and is given by

$$R_S = \frac{2GM}{c^2}$$

for a black hole of mass M.

(i). Assume that the two black holes begin to merge when the separation of their centres is equal to the sum of their Schwarzschild radii. Show that the highest frequency attained by the chirp is

$$f_c = \frac{\omega_c}{\pi} = \frac{c^3}{2\sqrt{2}\pi G(M+m)}.$$

Using the data from **Figure 5** estimate the individual masses M, m of the black holes and their sum $M + m$ in terms of Solar masses.

(j). Finally, estimate the total energy radiated during the collision, stating any necessary assumptions, expressed in terms of mass equivalent to Solar masses.

Qu 3. Phase Space Physics

In cases involving multiparticle systems, it can be an advantage to consider an abstract space called phase space. In a particle accelerator such as the LHC at CERN, managing to produce a finer and denser beam of particles at a discrete energy, reducing the phase space of the beam, has been one of the keys to its success. The approach known as stochastic cooling won Simon van der Meer a Nobel Prize in Physics.

In 1-D, for a moving particle, the phase space trajectory is a line on a graph. The line shows the momentum p_x, versus position, x, on a p_x versus x graph. For a free particle of mass m, moving at constant speed v_x in the positive x-direction, the trajectory in phase space as it passes through the origin is shown in the $p_x - x$ graph of **Fig. 6**. Included is an arrow on the trajectory to show its direction in time along the trajectory.

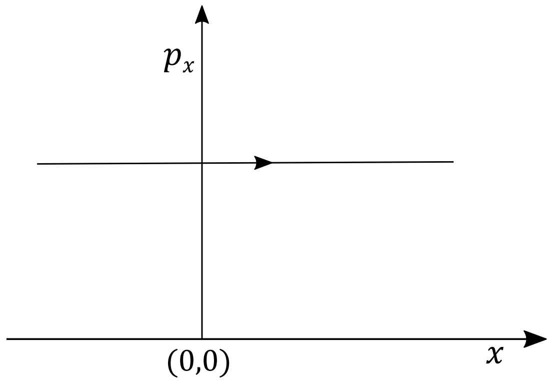


Figure 6

(a). By calculation, or otherwise, determine the behaviour of a particle with the trajectory in phase space given by the graph in **Fig. 7**.

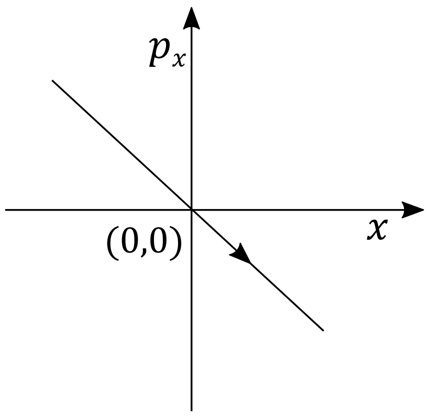


Figure 7

(b). A simple pendulum consists of a light, rigid rod of length ℓ, with mass m attached at the end, as shown in **Fig. 8**. It swings freely under gravity.

(i) Write down an expression for the total energy E of the pendulum in terms of p, the magnitude of the momentum and θ, the angle of displacement of the rod from the vertical.

(ii) For small amplitude oscillations, it executes simple harmonic motion (SHM). Using a small angle approximation for θ, and expressing the total energy in terms of angular variables p_θ and θ, where $p_\theta = \ell p$, sketch the phase space trajectory on a $p_\theta - \theta$ graph. Include arrows on the trajectory. Show the limits on your axes, in terms of m, E, ℓ, g.

(iii) On your graph, sketch a trajectory corresponding to a pendulum which has twice the initial amplitude of swing (but still SHM). How will this change the area of phase space enclosed by the trajectory?

(iv) If the pendulum loses 0.002% of its energy each swing, what will be the percentage area of phase space remaining enclosed by the trajectory after 400 swings?

(v) For the phase space trajectory of a simple harmonic oscillator, give a physical interpretation of the angles at which the trajectory crosses the horizontal and vertical axes.

(vi) If the pendulum has a small amount of viscous damping due to air resistance, what will be the trajectory in phase space as the pendulum slows down to very small amplitudes? Can it be said that the pendulum will ever stop swinging?

(vii) If the pendulum is damped by contact friction at the support, sketch the phase space trajectory as the pendulum reduces to very small amplitude swings.

(viii) If the pendulum has a large total energy, its motion will not be oscillatory but continuous rotation, with the kinetic energy term greater than the the potential energy term. Sketch a new $p_\theta - \theta$ graph which has trajectories for the pendulum (small KE term) and the circulating rod (large KE term).

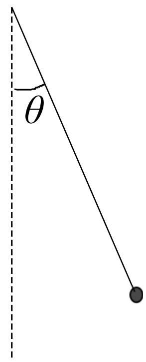


Figure 8

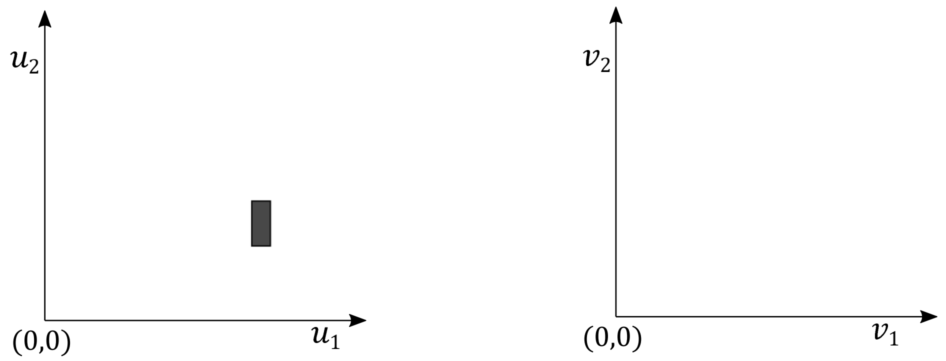
(c). A linear, elastic collision takes place between particles of mass m_1 and m_2, with initial velocities u_1 and u_2, resulting in final velocities of v_1 and v_2 respectively.

(i) Show that the velocities after collision can be expressed in terms of those before, in the form $v_1 = au_1 + bu_2$ and $v_2 = cu_1 + du_2$

(ii) Determine the coefficients a, b, c and d in terms of the masses m_1 and m_2.

(iii) If many repeated collisions are made using the same masses, but with various initial speeds in the ranges $u_1 + \Delta u_1$ and $u_2 + \Delta u_2$, then final speeds will lie in the ranges $v_1 + \Delta v_1$ and $v_2 + \Delta v_2$. On graph axes of u_2 against u_1, the velocity domain of the collisions are represented by the scatter of points within the rectangle, shown in **Fig. 9a**. If a graph of v_2 against v_1 is plotted, as in **Fig. 9b**, how does the velocity domain (the rectangular area of **Fig. 9a**) transform onto the new axes?

Hint: for two vectors $f = f_1\mathrm{X} + f_2\mathrm{Y}$ and $g = g_1\mathrm{X} + g_2\mathrm{Y}$ on $\mathrm{X} - \mathrm{Y}$ axes, which form the two sides of a triangle, the area of the triangle is given by the modulus (ignore any sign) of $\frac{1}{2}(f_1 g_2 - f_2 g_1)$.



(a) Velocity domain of particle velocities before collisions

(b) Velocity domain of particle velocities after collisions

Figure 9

(d). The time development of particles in phase space can be seen if a set of particles, each of mass m, have a range of momenta, and are distributed in the shaded, square region of phase space shown in the graph of **Fig. 10**. After a time t later, what will be the possible distribution of the particles in phase space if they do not interact? Sketch your result on a copy of the graph of p vs. x.

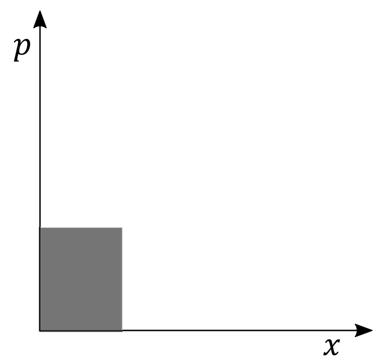


Figure 10

Qu 4. Solar Sail

It has been proposed that tiny probes could be sent to other stars by accelerating them away from Earth using a powerful laser.

(a). To push the probe hard enough for it to reach other solar systems will require a large laser focused on the sail of a small probe. One way of focusing the laser is to make use of existing optics by using a telescope in reverse. Given a telescope of diameter D emitting a laser beam of wavelength λ, and a probe sail of diameter d, what is the distance R the probe can get before less than 90% of the light hits the sail? How far is this for a sail of size $1\,\mathrm{m}$, a telescope diameter of $10\,\mathrm{m}$ and a laser wavelength of $600\,\mathrm{nm}$.

(b). Assume that when the monochromatic laser light hits the sail of the probe at right angles it has power W. What is the force F exerted on a black (perfectly absorbing) sail and a silvered (perfectly reflective) sail? What is another major change in using a reflective sail?

(c). Using your previous expression, find the time it would take a probe of mass m to travel a distance L. For simplification, you may assume that that the probe feels the full force of the laser beam up to distance R, and nothing after that, and that it has a perfectly reflective sail. The proposal calls for the probe to have mass of $10^{-3}\,\mathrm{kg}$, the laser power to be $100\,\mathrm{GW}$ and for the probe to go to Alpha Proxima which is 4.2 light years away. Using the optics described in the first part, how long will this take?

(d). One way of reducing this time considerably is to increase R by using a different optical set up. Rather than using one telescope, several telescopes could be used together in order to create an interference pattern. What is the maximum possible effective diameter of such a setup? What would be the travel time to Alpha Centauri under this setup? Comment on the validity of any extra approximations you had to make in this case.

Qu 5. Particle Physics

The Zero Momentum Frame (ZMF) is a frame of reference in which the total momentum of a collection of particles is zero.

(a). Consider two particles of masses m_1 and m_2 with velocities u_1 and u_2 along a line.

(i) Show that the addition of a velocity

$$u' = -\frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

to both particles transforms the system to a new frame of reference in which the total momentum is zero.

(ii) A particle of mass m, moving with velocity u in a straight line, collides elastically head on with an identical stationary particle. Using the ZMF, or otherwise, find the velocities of the two particles after the collision.

(iii) In a game of pool, the white cue ball collides elastically with a red ball. At the instant of collision, the cue ball is moving with speed u at an angle θ to the line connecting the centres of the two balls as shown in **Figure 11**. By transforming to a ZMF parallel to the line of centres, or otherwise, find the angle between their directions of motion after the collision. You may assume that the two pool balls are perfectly smooth.

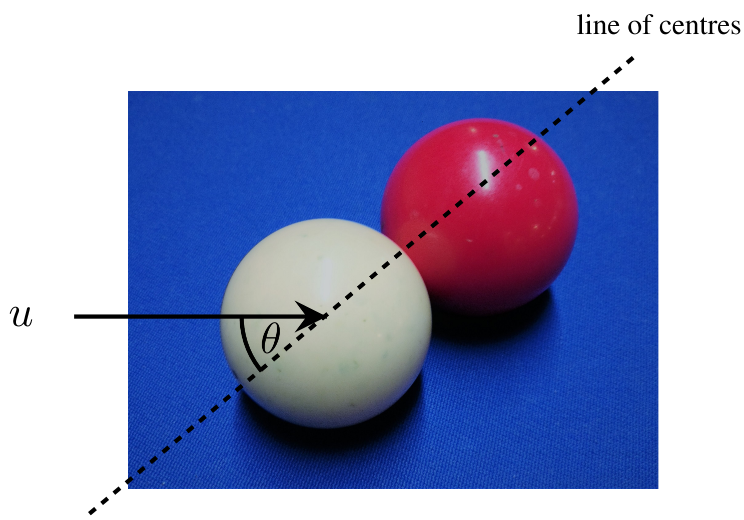


Figure 11: A collision between two pool balls.

(b). The energy and momentum of a particle travelling at speeds approaching the speed of light, c, may be written as

$$E = \gamma m c^2 \qquad \& \qquad p = \gamma m v$$

where m is the mass of the particle, v is its velocity and γ is the relativistic factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(i) Show that

$$E^2 = p^2 c^2 + m^2 c^4$$

and check that this gives the usual formulas both for massive particles at rest, and for massless particles.

(ii) When an electron annihilates with a positron (the antiparticle of an electron, having all the same properties as an electron but opposite charge), photons are produced. Explain why it is not possible for this process to produce a single photon only, and if exactly two photons are produced, how will they be emitted in the ZMF?

(iii) A positron moving with velocity v in the positive x-direction annihilates with a stationary electron, producing two identical photons. By using the ZMF, or otherwise, find the angle between the emitted photons in terms of $\gamma_v = (1 - v^2/c^2)^{-1/2}$. What is the angle in the limit $v \to 0$?

You may find the following information helpful: When transforming to a new frame of reference with velocity v' relative to the original frame, the momentum of a particle in the new frame, p', is related to the momentum p and energy E in the original frame by

$$p' = \gamma_{v'} \left(p + \frac{v'}{c^2} E \right)$$

where $\gamma_{v'} = (1 - v'^2/c^2)^{-1/2}$.

Hint: In the ZMF consider the photons to be emitted parallel to the y-direction.

(iv) Fluorine-18 undergoes beta-plus decay and is commonly used in positron emission tomography. Positrons are emitted with a range of **kinetic** energies up to a maximum of $635\,\mathrm{keV}$. If a positron were to immediately annihilate with an electron, producing two photons, what would be the minimum angle between them?

(v) Pair production is the process by which a photon may interact with a nucleus to produce an electron positron pair. Show that this process will not occur in free space, *i.e.* that the process depicted in **Figure 12** is forbidden.

Hint: Consider the relationship between Energy and Momentum for the photon.

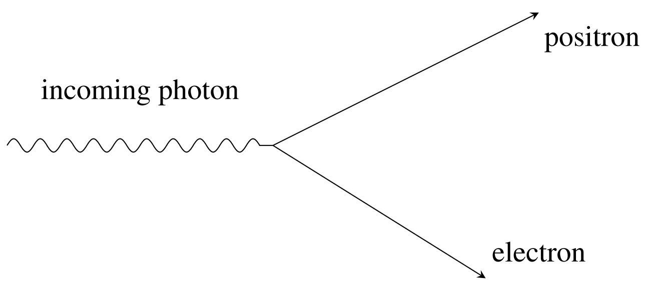


Figure 12: Pair production in free space.

END OF PAPER

Questions proposed by:
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