This is a guide and is not the final version for marking scripts

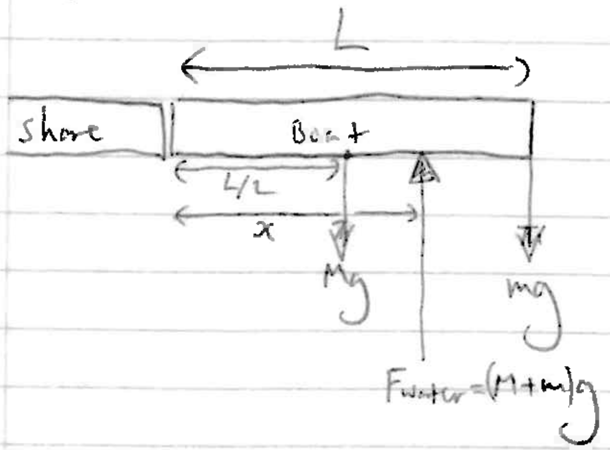
Qu 1 / 25

~~BPhO Round 2 2018-19~~

Q1) (a) (i) No external forces act on boat + motor + instructor horizontally. 'System' must therefore remain at rest / moving with const. velocity at end depending on the start. Assume at rest to start. This means that centre of mass of system must remain at rest. ✓

Let M = mass of boat & m = mass of instructor + motor.

Before:

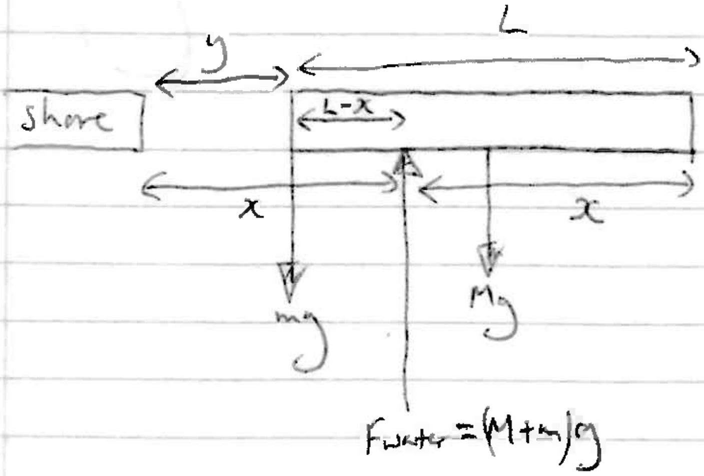


Taking moments about shore:

$$(M+m)gx + Mg\frac{L}{2} = mgL$$

$$x = \frac{\left(\frac{M}{2}+m\right)}{M+m}L$$ ✓

After:



$$y = x - (L - x)$$

$$= 2x - L$$

$$= \frac{M+2m}{M+m}L - L$$

$$= \frac{M+2m-M-m}{M+m}L$$

$$= \frac{m}{M+m}L$$ ✓

This gives $y = \frac{(60+40)}{180+(60+40)} \times 5\,m$

$= \frac{25}{14}\,m$

$\approx 1.79\,m$

(ii) Now $F_{ext\text{-}horizontal} \propto v$. Assuming starts and ends at rest:

$\Delta(mv) = 0$

$\Rightarrow \int F\,dt = 0$

$\Rightarrow \int v\,dt = 0$

$\Rightarrow$ displacement $= 0$.

So instructor will be at shore.

for a suitable model.

5

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(ii) Now $F_{ext\text{-}horizontal} \propto V$. Assuming starts and ends at rest:

$\Delta(mv) = 0$

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for a suitable model.

5

2019 BPhO Qns and Solutions

Short Qn

The BepiColumbo mission to Mercury launched in October 2018 is only the second satellite destined to enter an orbit around Mercury, 6 years after launch in 2024. Given that, at their closest, Mercury and Mars are approximately equidistant from the Earth, explain why a mission to put a satellite in orbit around Mercury requires significantly more energy expenditure than a similar mission to Mars.

Qu 1(b)

Answer

Looking for following points:

1) In order to orbit a planet it is necessary to achieve an appropriate velocity, firstly to match the speed of the planet's own orbit around the sun and then secondly to stay in orbit about the planet. ✓
2) Launching to Mars is "uphill" so the satellite needs to be provided with sufficient energy to gain the GPE needed and the KE on arrival.
3) Earth's orbit is faster than Mars's so there is no need to catch up with Mars, hence KE needed on arrival is minimal. ✓
4) Going to Mercury is "downhill" so apparently free but the GPE lost will end up as KE, leaving the satellite travelling too fast. Hence fuel must be burned to slow down the craft, lest it fall into the Sun. ✓
5) Mercury's gravitational field is relatively weak compared to the sun, so a lot of slowing is needed. Also Mercury's orbit is faster than Earth's so some boost is needed to catch it up.
6) (Main point) GPE varies as 1/r so change in GPE going from Earth up to Mars is much less than change from Earth to Mercury. ✓

If point 6 is not appreciated, marks must be very limited.

(4)

Long Question. Mechanical Toy

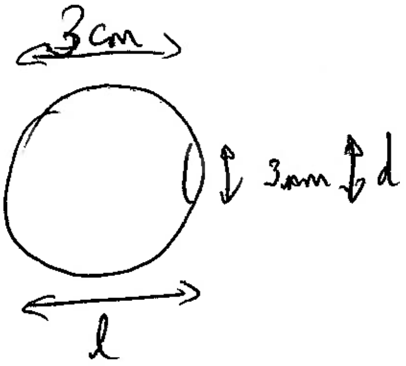
Two masses, m, are mounted at the end of symmetrical arms of length l each at an angle a to the body of the toy at a height h above the pivot point. Initially we shall take the body of the toy to be of zero mass.

Qu 1(c)

- The lunes are more stable when they are concave outwards than inwards. ✓
- When the ball is tapped, the flow of air out through the hole is small. ✓
- The small increase in pressure is enough to cause some lunes to bulge outwards. ✓
- Since this is a more stable state, generally the more lunes will expand outwards than be pushed inwards by tapping. ✓

(4)

Qu 1(d)



$\theta = (1.22)\lambda/d$

Diffraction limited spot.

Diameter of spot is $l\theta$

area is $\frac{\pi}{4}(l\theta)^2$

with $\theta = (1.22)\lambda/d$

$\therefore$ area $= \frac{\pi}{4} l^2 \cdot (1.22)^2 \frac{\lambda^2}{d^2}$

$\approx l^2 \frac{\lambda^2}{d^2}$

Initial ~~energy~~ power arriving is $I_0 \cdot \pi \frac{d^2}{4}$ (beam is similar to size of pupil).

$\therefore$ power on retina is $\dfrac{I_0 \frac{\pi d^2}{4}}{l^2 \lambda^2 / d^2}$

$= I_0 \frac{\pi}{4} \cdot \frac{d^4}{l^2 \lambda^2}$

$= 1\times10^{-3} \times \frac{(4\times10^{-3})^2}{(3\times10^{-2})^2(500\times10^{-9})^2}$

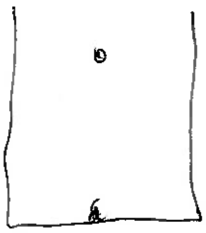
$= \approx 10^8 \, W/m^2$

Initial intensity $\approx 10^2 \, W/m^2$

So factor of 10^6 gain in intensity

(5)

Qu 1 (e)



One proton at the bottom

Balance the weight of a proton with electrostatic repulsion ✓

$$mg = \frac{1}{4\pi\varepsilon_0} \cdot \frac{e^2}{r^2}$$

$$1.67\times10^{-27}\times10 = 9\times10^{9}\times\frac{(1.6\times10^{-19})^2}{r^2}$$

$$\therefore\ r^2 = \frac{9\times10^{9}\times(1.6\times10^{-19})^2}{1.67\times10^{-27}\times10}$$

$$= \frac{3}{5}\times\frac{10^{10}}{10^{-26}}\times(1.6\times10^{-19})^2$$

$$r = 0.8\times10^{18}\times1.6\times10^{-19}$$

$$= 1.3\times10^{-1}$$

$$= 13\ \text{cm}$$

✓

So 1 or 2 protons (approx). ✓

(3)

Qu. 1(f).

$0.78\,m^3$ of N_2 , $M = 28.0$ g/mole

$0.21\,m^3$ O_2 , $M = 32.0$ g/mole

$0.01\,m^3$ A , $M = 39.9$ g/mole

Gases at same temperature and pressure to occupy same volume per mole

Average molar mass of air is $M = 0.78 \times 14.0 \times 2$
$+ 0.21 \times 16.0 \times 2$
$+ 0.01 \times 39.9$

$M = 28.96$ g/mole.

Mass of $1\,m^3$ is $\frac{1}{1.23} = 0.813$ kg
$= 813$ g

Hence no. of moles is $\frac{813}{28.96}$ moles

$= \underline{28.1 \text{ moles}}$

$KE = \frac{3}{2}kT$ per molecule

$= \frac{3}{2} \times n \times N_A \cdot kT$

$= \frac{3}{2} n RT$

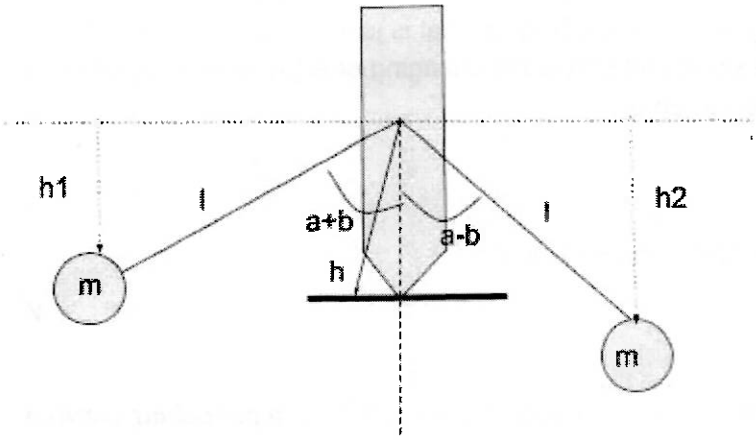
$= \frac{3}{2} \times 28.1 \times 8.31 \times 288$

$= \underline{\underline{101 \text{ kJ}}}$

(4).

Question 2 /25

Answer



a)

h_1 = h cos(b) -l cos(a+b) = h cos(b) - l cos(a)cos(b) + l sin(a)sib(b)

h_2 = h cos(b) -l cos(a-b) = h cos(b) - l cos(a)cos(b) - l sin(a)sib(b)

U = mg(h_1 + h_2) = 2mgh cos(b) - 2mgl cos(b)cos(a) = 2mg cos(b) (h-l cos(a))

b)

dU/db = -2mg sin(b) (h - l cos(a))

This is zero for b = 0,180,360... (0, π, 2π...)

Firstly only values in the domain 0-359 are physically valid as one rotation brings the system back to 0. This leaves 0 and 180. 0 fits with the diagram as drawn. b=180 also works mathematically so the only reasons to reject would be either to consider stability a little further (e.g. as in part c or by asserting that a small displacement from the equilibrium will result in the toy toppling, so it is not stable) or to compare to the diagram where we are clearly requiring the masses to be drawn below the horizontal, not above. (3)

c)

i) d^2U/db^2 must be positive as this implies the energy function is a minimum wrt b rather than a maximum, and hence the toy is stable as displacement would require it spontaneously to move "uphill"

ii) d^2U/db^2 = -2mg cos(b) (h-l cos(a))

For this to be positive with b=0, (h - l cos(a)) must be negative

I.e. l cos(a) > h

But l cos(a) is the vertical height of the mass below the attachment point and h is the vertical height of the pivot below the attachment point ie the condition is that the masses must be below the pivot point. (4)

d)

GPE = 2mg cos(b) (h-l cos(a)) = A cos (b) where A = 2mg(h-l cos (a)) is constant

Using small angle approx cos (b) = 1-b^2/2 so GPE = A(1-x^2/2h^2) using the approximation for b.

For the KE we need to find the speed of each mass, which is (l sin(a)) db/dt = l sin(a) $\dot{x}$/h again using the approximation for b and (l sin(a)) as the radial distance of the masses from the pivot point (ie using v = rw). Hence KE = 2 x ½ m (l sin(a) $\dot{x}$/h)^2

(4)

e) A mass on a spring has EPE = ½ kx^2 and KE = ½ m $\dot{x}^2$. Angular frequency is given by square root of the ratio of the constant multiplying x and that multiplying $\dot{x}$. Looking at our KE and PE eqns for the teeter toy we similarly have a constant multiplying each of these (we can ignore the constant energy term A in the GPE as that is just due to the choice of zero position - for the dynamics the significant part is the changing energy) and so by analogy the angular frequency should be given by:

$$\omega = \sqrt{\frac{-A/h^2}{2m(l\sin a/h)^2}}$$

The minus sign is because our GPE term is -½ A x^2

$$\omega = \sqrt{\frac{g(l\cos a - h)}{(l\sin a)^2}}$$

(4)

(after cancelling m,2, h etc). Which looks an awful lot like $\sqrt{g/l}$ for a pendulum, within a geometric factor or two.

f) Addition of the Mgh term for the mass of the support changes the condition for the second derivative to be negatiive to

2mg(h-l cos(a)) + Mgh <0

This is equivalent to the centre of mass of the system being more than h below the attachment point, ie below the point of contact.

In terms of the new oscillation frequency we have the additional GPE term Mgh cos(b) so A is now 2mg(l cos (a) - h) + Mgh. We also have the additional KE term ½ M$\dot{x}^2$ as the centre of mass of the toy moves with speed $\dot{x}$. Hence our expression becomes (using the square to avoid the large square root):

$$\omega^2 = \frac{2mg(l\cos a - h) + Mgh}{2ml^2\sin^2 a + Mh^2}$$

(the h^2 which previously cancelled throughout now does not as the KE for the body of the toy does not contain that term).

If we write the original expression for ω^2 as P/Q where P = 2mg(lcos(a) -h) and Q = 2m(lsin(a))^2 then the new expression is (P+R)/(Q+S) with R = Mgh and S = Mh^2

If the new oscillation is faster we expect

(P+R)/(Q+S) > P/Q

Ie

PQ + RQ> PQ + PS

RQ>PS

Mgh 2m(lsin(a))^2 > Mh^2 2mg(lcos(a) -h)

(l sin (a))^2 > h(l cos(a) - h)

(5).

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PQ + RQ> PQ + PS (5).

RQ>PS

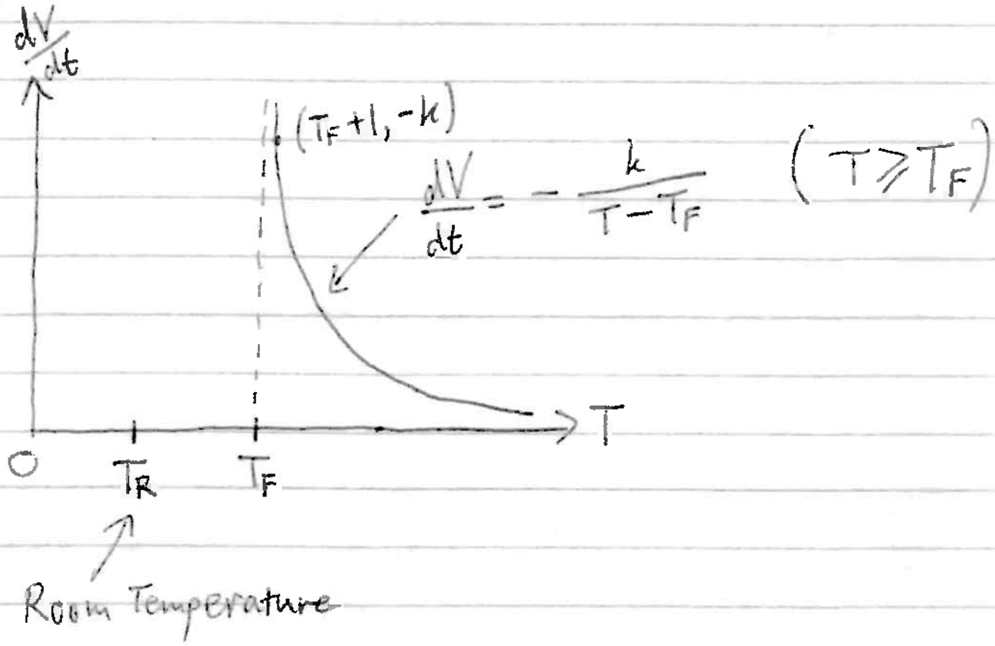
Mgh 2m(lsin(a))^2 > Mh^2 2mg(lcos(a) -h)

(l sin (a))^2 > h(l cos(a) - h)

BPhO Round 2 2018-19

Q3 Sisyphus's Coffee

(a) Sketch:



As can be seen in the sketch, as the coffee cools and its temperature decreases, the rate at which it can be drunk increases.

N.B. $[k] = \left[\frac{dV}{dt}\right][T - T_F] = L^3 T^{-1}$ (H)

Consider a point on the curve where $T = T_F + 1$ (i.e. "just" above the critical temperature, T_F, which is itself above T_R). Here, $\frac{dV}{dt} = -k$. With a 'typical' mug of diameter ≈ height ≈ 8cm, a full mug of liquid at a pleasant temperature may perhaps be consumed in a time of about 15s. This gives

$$\frac{dV}{dt} = \frac{-\pi \times (4\,\text{cm})^2 \times 8\,\text{cm}}{15\,\text{s}} \simeq -30\,\text{cm}^3\,\text{s}^{-1}$$

i.e. $\underline{\underline{k \simeq 30\,\text{cm}^3\,\text{K}\,\text{s}^{-1}}}$ (Any reasonable estimate)

②

(b) Expect heat loss via:

- Conduction through walls then radiation to surroundings & conduction to surrounding air followed by convection through air.

$\overbrace{\hspace{6cm}}^{\text{Assumed small}}$

- Radiation from exposed surface, evaporation from exposed surface & conduction to air followed by convection through air.

Mug has two walls filled with foam, which will contain trapped air, a poor conductor. Therefore expect dominant mode of heat loss to be through exposed surface. Expect rate of heat loss to depend on (cross-sectional) area of surface and temperature difference between coffee and surroundings.

from point of view of mug

Simplest reasonable difference would be linear (the greater the area the greater the heat loss rate & the greater the temperature difference the greater the rate of heat loss), so expect:

$$\frac{dQ}{dt} \propto A \quad \& \quad \frac{dQ}{dt} \propto (T - T_R)$$

$$\Rightarrow \quad \frac{dQ}{dt} = -hA(T - T_R) \qquad (- \text{ since decreasing rate})$$

Using $\delta Q = mc\delta T$, i.e. $\frac{dQ}{dT} = mc$ (m = mass of coffee, c = specific heat capacity of water)

$$\Rightarrow \frac{\delta Q}{\delta t} = mc\frac{\delta T}{\delta t} \quad \text{or} \quad \frac{dQ}{dt} = mc\frac{dT}{dt}$$

$\therefore$ Rate of cooling is $\frac{dT}{dt} = -\frac{hA}{mc}(T - T_R)$

or $$\frac{dT}{dt} = -\frac{hA}{\rho c}\frac{(T - T_R)}{V}$$

A = cross-sectional area of mug
ρ = density of water
V = volume of coffee
T = temp. of coffee

N.B. this is Newton's law of cooling which is true for conduction, and approximately true for convection/evaporation, but definitely not true for radiation, for which expect $P \propto T^4$. Therefore ignore radiative losses in this scenario.

(c) Starting with $\frac{dV}{dt} = -\frac{k}{(T - T_F)}$

Shorthand $\frac{dV}{dt} = \dot{V}$; $\frac{dT}{dt} = \dot{T}$
$\frac{d^2V}{dt^2} = \ddot{V}$

$$\Rightarrow \frac{d^2V}{dt^2} = \frac{k}{(T - T_F)^2}\frac{dT}{dt}$$

$$= \frac{1}{k}\left(\frac{dV}{dt}\right)^2\frac{dT}{dt} \quad \text{using } \frac{1}{T - T_F} = -\frac{1}{k}\frac{dV}{dt}$$

$$= -\frac{hA}{\rho c k}\frac{1}{V}\left(\frac{dV}{dt}\right)^2(T - T_R) \quad \text{using } \frac{dT}{dt} = -\frac{hA}{\rho c}\frac{(T - T_R)}{V}$$

$$= -\frac{hA}{\rho c k}\frac{\dot{V}^2}{V}\left(-\frac{k}{\dot{V}} + T_F - T_R\right) \quad \text{using } \dot{V} = \frac{-k}{T - T_F}$$

$$\ddot{V} = +\frac{hA}{\rho c}\frac{\dot{V}}{V} - \frac{hA}{\rho c k}(T_F - T_R)\frac{\dot{V}^2}{V}$$

Using suggested relationship $\ddot{V} = \dot{V}\frac{d\dot{V}}{dV}$ & let $m = \frac{hA}{\rho c}$

gives $\cancel{\dot{V}}\dfrac{d\dot{V}}{dV} = m\dfrac{\cancel{\dot{V}}}{V} - \dfrac{m}{k}(T_F - T_R)\dfrac{\dot{V}^{\cancel{2}}}{V}$

$$\Rightarrow \frac{d\dot{V}}{dV} = \frac{m}{V}\left(1 - \frac{T_F - T_R}{k}\dot{V}\right)$$

$$\Rightarrow \int_{\dot{V}_0}^{\dot{V}} \frac{d\dot{V}}{1-\alpha\dot{V}} = \int_{V_0}^{V} \frac{m\,dV}{V} \qquad \left(\text{with } \alpha = \frac{T_F - T_R}{k}\right)$$

$$\Rightarrow \left[-\frac{1}{\alpha}\ln(1-\alpha\dot{V})\right]_{\dot{V}_0}^{\dot{V}} = m\ln\left(\frac{V}{V_0}\right)$$

$$\Rightarrow -\frac{1}{\alpha}\ln\left(\frac{1-\alpha\dot{V}}{1-\alpha\dot{V}_0}\right) = m\ln\left(\frac{V}{V_0}\right)$$

$$\Rightarrow \ln\left(\frac{V}{V_0}\right) = \ln\left[\left(\left(1 + \frac{(T_F - T_R)}{\cancel{k}}\cdot\frac{\cancel{k}}{T - T_F}\right)\left(1 + \frac{(T_F - T_R)}{\cancel{k}}\cdot\frac{\cancel{k}}{T_0 - T_F}\right)^{-1}\right)^{-\frac{1}{\alpha m}}\right]$$

$$= \ln\left[\left(\frac{T - \cancel{T_F} + \cancel{T_F} - T_R}{T - T_F}\cdot\frac{T_0 - T_F}{T_0 - \cancel{T_F} + \cancel{T_F} - T_R}\right)^{-\frac{1}{\alpha m}}\right]$$

$$\Rightarrow V = V_0\left(\frac{(T_0 - T_R)(T - T_F)}{(T_0 - T_F)(T - T_R)}\right)^{\frac{k}{m(T_F - T_R)}} \qquad \left(m = \frac{hA}{\rho c}\right)$$

/25

Q4 Stable Nuclei

(a) (i) Binding energy = Energy released on formation of nucleus from free nucleons

$= \Delta m c^2$ (given sign)

where Δm = mass of free nucleons − mass of nucleus

$$\Rightarrow BE(A,Z) = \left[(A-Z)m_n + Zm_p - M(A,Z)\right]c^2$$
$$= a_1 A - a_2 A^{2/3} - a_3 Z^2 A^{-1/3} - a_4 (A-2Z)^2 A^{-1} - a_5 \delta(A,Z)$$

where a_1 to a_5 have been redefined as $a_1 = 15.8$ MeV etc. (i.e. factor of c^2 absorbed into units).

Recalling that the density of nuclei is found to be roughly constant, implying nuclear radius $r \propto A^{1/3}$ $(r = r_0 A^{1/3})$

- Nuclear volume* $\propto r^3$ so vol. $\propto A$, a_1 is therefore 'volume' term, depending simply on the number of nucleons. The greater the no. of nucleons, the greater the energy needed to separate them (or equivalently the greater the energy release when they form the nucleus) so the greater the BE.

* assumed spherical

②

- Surface area of nucleus (again assumed spherical) $\propto r^2$ and since $r \propto A^{1/3} \Rightarrow \underline{SA \propto A^{2/3}}$, $\underline{a_2}$ is therefore surface term. Nucleons at surface do not require as much energy to be separated from nucleus as nucleons within nucleus. This term therefore reduces the energy required to separate the nucleus into its nucleons so the a_2 term is subtracted from the a_1 term. The Larger the surface area of the nucleus, the greater the no. of nucleons at surface and the more the BE is reduced as a result. ✓✓

- Coulomb repulsion of protons in nucleus means that they effectively assist the separation of the nucleus into its constituent nucleons thus providing another term which reduces the amount of energy required to separate the nucleus into its nucleons. This leads to another term which is subtracted from the a_1 term. The electrostatic energy of a distribution of charge Q with radius r is (as per the question) $E \propto Q^2/r$, and since $Q \propto Z$ and $r \propto A^{1/3}$ this is $E \propto Z^2 A^{-1/3}$ ✓ so this is the term $\underline{a_3}$. In fact, numerically. ✓✓

$$E = \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 r}$$

$$= \frac{3}{5}\cdot\frac{(Ze)^2}{4\pi\varepsilon_0 r_0 A^{1/3}}$$

$$= \frac{3e^2}{20\pi\varepsilon_0 r_0} Z^2 A^{-1/3}$$

Here the pre-factor $\frac{3e^2}{20\pi\varepsilon_0 r_0} = \frac{3\times(1.60\times10^{-19}\,C)}{20\times\pi\times(8.85\times10^{-12}\,Fm^{-1})\times1.2\times10^{-15}\,m} \times 10^{-6}\,MeV$

$= 0.719\ MeV$ ✓

Which is comparable with $a_3 = 0.714$ MeV

- It is known that many stable nuclei have roughly the same numbers of protons and neutrons, i.e. $\underline{Z = N = \frac{A}{2}}$. This is particularly true at lower mass numbers*. Such nuclei would have a larger binding energy than others as being more stable means a larger amount of energy is required to separate it into its individual nucleons. Vice-versa, nuclei without $Z = N = \frac{A}{2}$ will have a smaller binding energy. The $\underline{a_4}$ term takes this into account, being something which is subtracted when $Z \neq \frac{A}{2}$ and being zero when $Z = A/2$*. This is the so-called 'asymmetry term'.

(ii) Recalling that electric charge is present in the nucleus in discrete units of e, with a total of Ze for Z protons present, it is clear that the treatment of the nucleus as a continuous distribution of charge cannot be completely correct. In particular, if the nucleus has only 1 proton then it has nothing to repel so coulomb repulsion would not be present to reduce the binding energy. This can be accounted for by replacing $Z^2 \to Z(Z-1)$ in the a_3 term.

* The A^{-1} in the a_4 term ensures that it is less significant for higher mass numbers.

④

(b) (i) For a given (constant) A, the BE may be written in terms of Z as:

$$BE = -\underbrace{\left(a_3 A^{-1/3} + 4a_4 A^{-1}\right)}_{+ve} Z^2 + \underbrace{4a_4}_{+ve} Z + \underbrace{(a_1 - a_4) - a_2 A^{2/3} - a_5 \delta(A,Z)}_{\text{Independent of } Z}$$

Where the a_4 term has been expanded and terms have been re-grouped in powers of Z. As can be seen this is quadratic in Z, with a negative coefficient of Z^2 so will form an inverted parabola.

For Z, A > 0 (which they must be):

$$a_1 - a_4 = -7.4 \text{ MeV} < 0$$

$$-a_2 A^{2/3} = -18.3 A^{2/3} \text{ MeV} < 0$$

$$-a_5 \delta(A,Z) = \begin{cases} +12A^{-1/2} > 0 & \text{even, even} \\ -12A^{-1/2} < 0 & \text{even, odd} \\ 0 & \text{odd} \end{cases}$$

As $A^{-1/2} \leq 1 \Rightarrow 0 < 12A^{-1/2} \leq 12$

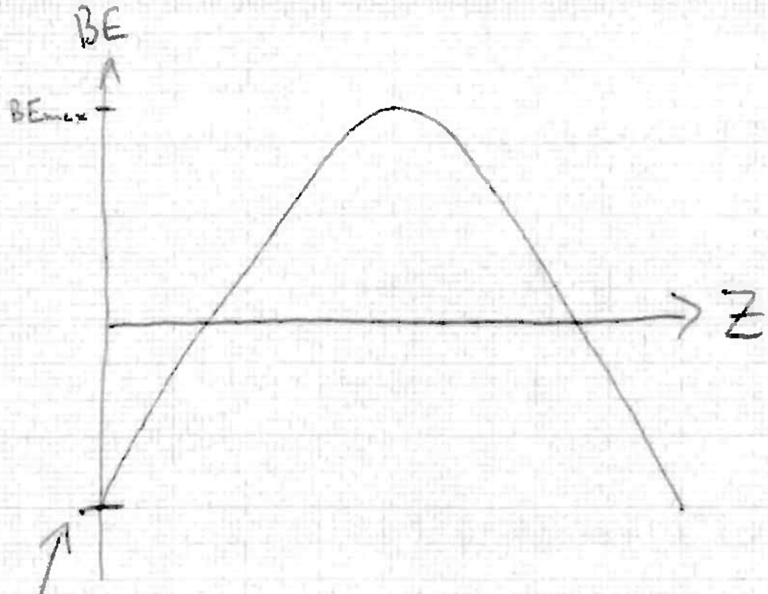
$A^{1/2} \geq 1 \Rightarrow -12A^{1/2} \leq -12$

$A^{2/3} \geq 1 \Rightarrow -18.3A^{2/3} \leq -18.3$

$\therefore (a_1 - a_4) - a_2 A^{2/3} \leq -25.7 \text{ MeV}$

So $(a_1 - a_4) - a_2 A^{2/3} - a_5 \delta(A,Z) < 0$

∴ Sketch



$-(a_4-a_1)-a_2A^{2/3}-a_5\delta(A,Z)$

Roots at "$\dfrac{-b\pm\sqrt{(b^2-4ac)}}{2a} = Z$"

where $b = 4a_4$, $a = -(a_3A^{1/3}+4a_4A^{-1})$

$c = -(a_4-a_1)-a_2A^{2/3}-a_5\delta(A,Z)$

The most stable nuclei will have the highest BE as they require the most energy to separate their nucleons from each other. For a given (i.e, constant) mass number, A, therefore need to find Z corresponding to BE_{max}. BE_{max} occurs when

$$\frac{dBE}{dZ} = 0 \quad ✓$$

$$\Rightarrow -2(a_3A^{1/3}+4a_4A^{-1})Z + 4a_4 = 0.$$

(Note: $\delta(A,Z)$ is not an explicit function of Z so $\frac{d}{dZ}\delta(A,Z)=0$)

$$\Rightarrow Z = \frac{A}{2}\cdot\frac{1}{1+\frac{a_3}{4a_4}A^{2/3}} \quad ✓$$

(c)

Now $\frac{a_3}{4a_4} \simeq 0.008 \sim O(10^{-2})$

$$A^{2/3} \sim O(10^0) \text{ if } A \text{ is } O(10)$$

$$\sim O(10) \text{ if } A \text{ is } O(100)$$

$\therefore \frac{a_3}{4a_4} A^{2/3} \sim O(10^{-2})$ to $O(10^{-1})$

So $\frac{a_3}{4a_4} A^{2/3}$ can be reasonably well treated as 'small', certainly < 1

& $\frac{1}{1 + \frac{a_3}{4a_4} A^{2/3}} \simeq 1$ in a basic approximation, so $\underline{\underline{Z \simeq \frac{A}{2}}}$

(ii) Binding energy per nucleon:

$$BE/A = a_1 - a_2 A^{-1/3} - a_3 Z^2 A^{-4/3} - a_4 (A - 2Z)^2 A^{-2} - a_5 \delta(A,Z) A^{-1}$$

If A is odd then $\delta(A,Z) = 0$.

As can be seen from fig. 5 BE/A has a maximum as a function of A. With $Z = \frac{A}{2}$ therefore maximise $\frac{BE}{A}$ w.r.t. A. Most stable nucleus is at peak of BE/A. With $Z = \frac{A}{2}$, a_4 term vanishes so:

$$BE/A = a_1 - a_2 A^{-1/3} - \frac{a_3}{4} A^{2/3}$$

$$\frac{d}{dA}\left(\frac{BE}{A}\right) = \frac{1}{3} a_2 A^{-4/3} - \frac{1}{6} a_3 A^{-1/3}$$

So $\frac{d}{dA}\left(\frac{BE}{A}\right) = 0$ gives $2a_2 - a_3 A = 0 \quad (A \neq 0)$

So $A = \frac{2a_2}{a_3} = 51.4$

i.e. predicted mass number of most stable nuclei is about

$$A \approx 51$$

N.B. About right given that we know max is around Fe-56

(c) Analogy of $E = \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 r}$ for gravity would be

$$E_{grav} = -\frac{3}{5}\frac{GM^2}{r}$$

attractive rather than repulsive

With $m_p \approx m_n =$ "m" (as in data sheet at front)*, $M = Am$

and $r = r_0 A^{1/3}$ so

$$E_{grav} = -\frac{3}{5}\frac{Gm^2}{r_0}A^{5/3} \longrightarrow +\frac{3}{5}\frac{Gm^2}{r_0}\frac{A^{5/3}}{A} \text{ in } \frac{BE}{A}$$

Continue with A odd so $\delta(A,Z) = 0$, a neutron star has

$Z = 0$ so:

$$\frac{BE}{A} = a_1 - a_2 A^{-1/3} - a_4 + \frac{3}{5}\frac{Gm^2}{r_0}A^{2/3}$$

* difference between this and, e.g. use of unified atomic mass unit negligible in this basic estimation.

Now: $A \gg 1$ so a_2 term will be negligible.

A nucleus (neutron star) will just be stable if $BE = 0$

(and hence $BE/A = 0$)

$$\Rightarrow A^{2/3} = (a_4 - a_1) \times \frac{5 r_0}{3 G m^2}$$

$$\Rightarrow A^{2/3} = (23.2 - 15.8) \times 10^{6} \times 1.6 \times 10^{-19} \times \frac{5 \times 1.2 \times 10^{-15}}{3 \times 6.67 \times 10^{-11} \times (1.67 \times 10^{-27})^2}$$

$$\Rightarrow A^{2/3} = \frac{(23.2 - 15.8) \times 1.6 \times 1.2 \times 5}{3 \times 6.67 \times (1.67)^2} \times \frac{10^{6} \times 10^{-19} \times 10^{-15}}{10^{-11} \times (10^{-27})^2}$$

$$\simeq 1.3 \times 10^{37}$$

$$\Rightarrow A \simeq 1.4 \times 10^{55.5}$$

$$A \simeq 4.5 \times 10^{55}$$

This leads to $M = Am$

$$\approx 7.6 \times 10^{28} \text{ kg}$$

$$M \simeq 10^{29} \text{ kg}$$

(d) Recall that $Z = \frac{A}{2} \frac{1}{1 + \frac{a_3}{4a_4} A^{2/3}} = \frac{A}{2}\left(1 + \frac{a_3}{4a_4} A^{2/3}\right)^{-1}$

A better approximation of $\left(1 + \frac{a_3}{4a_4} A^{2/3}\right)^{-1} \simeq 1 - \frac{a_3}{4a_4} A^{2/3}$

so $Z \simeq \frac{A}{2}\left(1 - \frac{a_3}{4a_4} A^{2/3}\right)$

So considering $\underbrace{\text{A odd again}}_{S(A,Z)=0}$, we would have:

$$\frac{BE}{A} = a_1 - a_2 A^{-1/3} - a_3 \cdot \frac{1}{4} A^2 \left(1 - \frac{a_3}{4a_4} A^{2/3}\right)^2 A^{-4/3}$$

$$- a_4 \left(A - A + \frac{a_3}{4a_4} A^{5/3}\right)^2 A^{-2}$$

$$= a_1 - a_2 A^{-1/3} - \frac{1}{4} a_3 A^{2/3} + \frac{1}{16} \frac{a_3^2}{a_4} A^{4/3} - \frac{1}{64} \frac{a_3^3}{a_4^2} A^2$$

Let $g(A) = \frac{BE}{A}$

$$\Rightarrow \frac{dg}{dA} = \frac{1}{3} a_2 A^{-4/3} - \frac{1}{6} a_3 A^{-1/3} + \frac{1}{12} \frac{a_3^2}{a_4} A^{1/3} - \frac{1}{32} \frac{a_3^3}{a_4^2} A = f(A)$$

$$\Rightarrow \frac{d^2 g}{dA^2} = -\frac{4}{9} a_2 A^{-7/3} + \frac{1}{18} a_3 A^{-4/3} + \frac{1}{36} \frac{a_3^2}{a_4} A^{-2/3} - \frac{1}{32} \frac{a_3^3}{a_4^2} = f'(A)$$

Noting that

$a_1 = 15.8$ MeV

$a_2 = 18.3$ MeV

$a_3 = 0.714$ MeV

$\frac{a_3^2}{a_4} = 0.0220$ MeV

$\frac{a_3^3}{a_4^2} = 6.76 \times 10^{-4}$ MeV

$$\frac{d^3 g}{dA^3} = \frac{28}{27} a_2 A^{-10/3} - \frac{2}{27} a_3 A^{-7/3} - \frac{1}{54} \frac{a_3^2}{a_4} A^{-5/3}$$

$$= f''(A)$$

$\underbrace{\text{use of Halley's method}}_{\text{for } f(A)=0}$ quickly converges on $A = 64$

for any reasonable starting point

Examples:

Initial A value	1st iteration	2nd iteration
50	64.0	64.2
60	64.2	64.2
70	64.2	64.2
40	63.0	64.2
30	60.0	64.2
51	64.1	64.2

For a decent initial value, e.g. Z close to 51, (the result of (b)(iii)), only 1 iteration is necessary.