BPhO
British Physics Olympiad

British Physics Olympiad 2019-20

Round 2 Competition Paper

Monday 27th January 2020

Instructions

Time: 3 hours (approximately 30 minutes for Q1 and 50 mins each for Qs 2, 3, 4).

Questions: All four questions should be attempted.

Marks: Questions 2, 3, 4 carry similar marks.

Solutions: Answers and calculations are to be written on loose paper. Graph paper should be made available. Students should ensure that their **name** and their **school/college** are clearly written on each and every answer sheet. Number each question clearly and number the pages. **EACH QUESTION ANSWERED must be started on a new page.**

Clarity: Solutions must be written legibly, in black pen (the papers are photocopied), and working down the page. Working, calculations, explanations and diagrams, properly laid out, must be shown for full credit. Scribble will not be marked and overall clarity is an important aspect of this exam paper.

Formula sheet: You are allowed any standard exam board data/formula sheet.

Calculators: Any standard calculator may be used, but calculators cannot be programmable and must not have symbolic algebra capability.

Confidentiality: Do not discuss any aspect of the paper on the internet until 8 am Saturday 1st February.

THIS QUESTION PAPER MUST NOT BE PHOTOGRAPHED OR TAKEN OUT OF THE EXAM ROOM.

Training Dates and the IPhO (Vilnius, Lithuania, 17th to 26th July)

Following this round, fourteen students eligible to represent the UK at the International Physics Olympiad (IPhO) will be invited to attend the **Training Camp** *to be held in the Physics Department at the University of Oxford,* **(Friday 27th to Tuesday 31st March 2020)**. *Five students will be selected for further training. From May there will be mentoring by email to cover some topics and problems. There will be a weekend* **Experimental Training Camp in Oxford 15th – 17th May (Friday evening to Sunday afternoon)**, followed by a **Training Camp** (in Cambridge), **Thursday 25th June – Wed 1st July**, with a second **Experimental Training Camp** (in Cambridge) **Sunday 12th – Thurs 16th July.**

Important Constants

Constant	**Symbol**	**Value**
Speed of light in free space	c	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Elementary charge	e	$1.60 \times 10^{-19}\,\mathrm{C}$
Acceleration of free fall at Earth's surface	g	$9.81\,\mathrm{m\,s^{-2}}$
Permittivity of free space	ε_0	$8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Mass of an electron	m_e	$9.11 \times 10^{-31}\,\mathrm{kg}$
Mass of a neutron	m_n	$1.67 \times 10^{-27}\,\mathrm{kg}$
Mass of a proton	m_p	$1.67 \times 10^{-27}\,\mathrm{kg}$
atomic mass unit	u	$1.661 \times 10^{-27}\,\mathrm{kg}$
(1u is equivalent to 931.5 MeV)		
Radius of a nucleon	r_0	$1.2 \times 10^{-15}\,\mathrm{m}$
Planck's constant	h	$6.63 \times 10^{-34}\,\mathrm{J\,s}$
Gravitational constant	G	$6.67 \times 10^{-11}\,\mathrm{m^{3}\,kg^{-1}\,s^{-2}}$
Boltzmann constant	k_B	$1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Molar gas constant	R	$8.31\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Specific heat capacity of water	c_w	$4.19 \times 10^{3}\,\mathrm{J\,kg^{-1}\,K^{-1}}$
Mass of the Sun	M_S	$1.99 \times 10^{30}\,\mathrm{kg}$
Mass of the Earth	M_E	$5.97 \times 10^{24}\,\mathrm{kg}$
Radius of the Earth	R_E	$6.38 \times 10^{6}\,\mathrm{m}$

$$T_{(\mathrm{K})} = T_{(^\circ\mathrm{C})} + 273$$

$$\text{Volume of a sphere} = \tfrac{4}{3}\pi r^3$$

$$\begin{aligned} e^x &\approx 1 + x + \ldots && x \ll 1 \\ (1+x)^n &\approx 1 + nx && x \ll 1 \\ \frac{1}{(1+x)^n} &\approx 1 - nx && x \ll 1 \\ \sin\theta &\approx \theta && \text{for } \theta \ll 1 \\ \tan\theta &\approx \theta && \text{for } \theta \ll 1 \\ \cos\theta &\approx 1 - \theta^2/2 && \text{for } \theta \ll 1 \end{aligned}$$

BLANK PAGE

Qu 1. General Questions

Write you answer starting on a new page for this question.

(a) For a modest temperature range, a sample of a metal in the form of a cube of side ℓ_0 and volume V_0 at room temperature is heated by ΔT. For small temperature changes, its volume expands by a temperature dependent factor such that $V = V_0(1 + \gamma\Delta T)$, whilst its length expands linearly by a factor given by $\ell = \ell_0(1+\alpha\Delta T)$. If the expansion factors are both small, $\gamma = k\alpha$, where k is a numerical factor. Derive a value for k.

(b) Two cells have e.m.f.s ε_1 and ε_2 and internal resistances r_1 and r_2.

(i) What are the effective e.m.f.s and internal resistances when these cells are arranged:

i. in series?
ii. in parallel? (assume that the e.m.f.s and internal resistances are now the same)

(ii) A cell has an e.m.f. of $1.0\,\mathrm{V}$ and an internal resistance of $50\,\Omega$. How many of these cells, and in what arrangement, would be needed to power a $5.0\,\mathrm{W}$, $6.0\,\mathrm{V}$ filament lamp?

(c) Shown in **Figure 1** below is an image of an electrified fence with ice crystals on it. Give a physics explanation of this effect that you can see.

Figure 1: credit: AAPT 2013 High School Photo Contest

(d) An electric car carries an accelerometer which also records the distance travelled. A graph of the recording when the car starts from rest is shown in **Figure 2**. Estimate the velocity of the car after five seconds, and the power per unit mass which is used to accelerate the car at that moment.

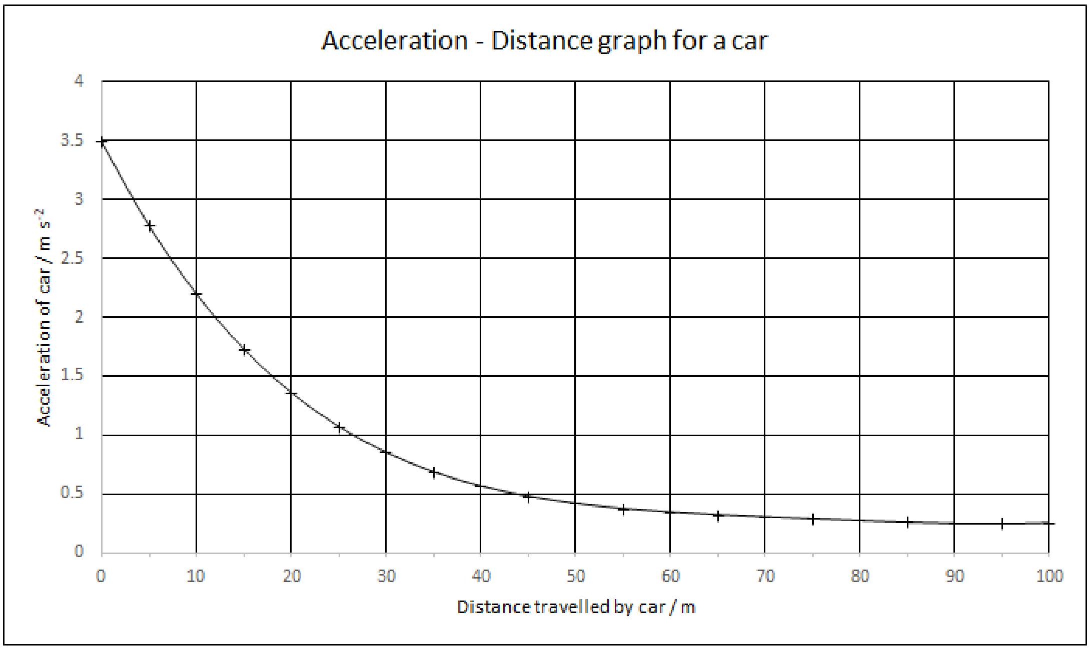

Figure 2

Qu 2. The Atmosphere & a Molecule

Write you answer starting on a new page for this question.

I The atmosphere

The atmosphere is composed of 78% nitrogen, 21% oxygen, 0.9% argon and about 0.1% trace gases. Water vapour varies between about 0%-4% depending on location and time of day.

(a) Consider a fluid at rest in a gravitational field g. Show that, if frictional forces within the fluid can be neglected, the pressure variation across a sufficiently small cube of incompressible fluid varies linearly with the height of the cube. This is referred to as hydrostatic pressure.

(b) In a simple model, the atmosphere is treated as an ideal gas.

- **(i)** Show that the variation in pressure with height is exponential. You may assume that the pressure variation across a very thin layer is hydrostatic, and that temperature is constant.
- **(ii)** Why is it safe to assume that g is constant too?
- **(iii)** What do you think is the biggest issue with the formula found in **(i)**?

(c) In another model, the gas (still ideal) is assumed to expand adiabatically as it rises. An adiabatic change is one for which the pressure, P, of the gas and its volume, V, are related by $PV^{\gamma} = \text{constant}$, where $\gamma = \frac{C_P}{C_V}$ with C_P and C_V the heat capacities of the gas per mole at constant pressure and volume respectively.

- **(i)** Find the rate of change of temperature with height, $\frac{\mathrm{d}T}{\mathrm{d}h}$ – known as the *temperature lapse rate* – under these circumstances. You may assume without proof that $C_P = C_V + R$ with R the molar gas constant.
- **(ii)** Hence or otherwise find a more realistic formula for the pressure variation with height in the atmosphere.

II Spectra of molecules

A diatomic molecule can store energy in the form of vibration and rotation of the molecule. The energies can be shown to be quantised i.e. there are discrete energy levels similar to the energy levels encountered in the visible spectrum of an atom.

(d) An HCl molecule can be modelled as two masses, m_1 and m_2 with mean separation ℓ at the ends of a stiff spring, with spring constant k, representing the bond. The frequency of vibration can be expressed in terms of k and μ, where $\mu(m_1, m_2)$ is function of m_1 and m_2. μ has dimensions of mass and is known as the reduced mass. By considering the centre of mass of the system or otherwise, derive an expression for the frequency of vibration f of the masses on the spring in terms of m_1 and m_2 and k. Hence write down the expression for μ.

The IR spectrum of HCl is shown in **Fig. 3**. The *wavenumber* on the horizontal axis is $1/\lambda$ measured in cm^{-1}. A low pressure gas of HCl is illuminated with IR radiation and the absorption measured over a range of frequencies. The HCl molecule will rotate corresponding to a number of close, discrete energy levels designated by the letter $J = 0, 1, 2, 3\ldots$ shown in **Fig. 4**. The two vibrational energy levels labelled with $n = 0$ and $n = 1$ correspond to a very much larger energy level difference.

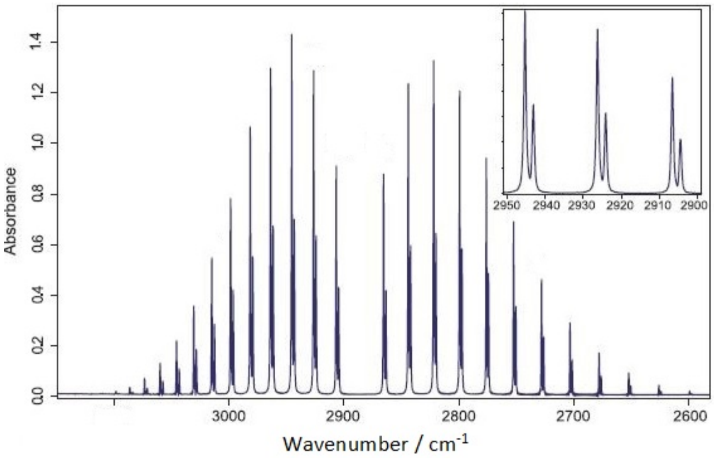

Figure 3: IR spectrum of HCl. The inset shows the effect of the Cl-35 and Cl-37 isotopic mass difference. Credit Azo Materials https://www.azom.com/article.aspx?ArticleID=15226

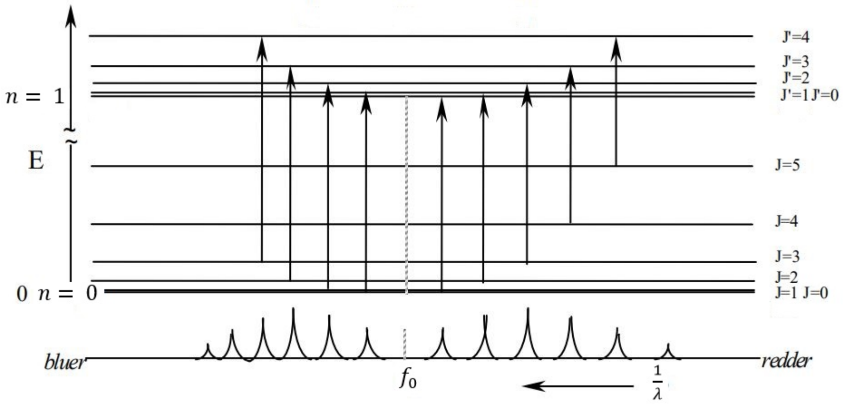

Figure 4: Energy level diagram for HCl. Two vibrational energy levels are shown ($n = 0$ and $n = 1$). The rotational energy levels are designated by the J values, with the $J = 0$ and $J = 1$ levels very close together and shown as a thick line. Only transitions corresponding to $\Delta J = \pm 1$ are allowed. Credit https://www.colby.edu/chemistry/PChem/lab/VibRotHClDCl.pdf; Vibration- Rotation Spectroscopy of HCl and DCl

(e) For the wavenumber value $2900\,\mathrm{cm}^{-1}$ calculate the energy E of the transition and calculate the corresponding temperature using $E = k_{\mathrm{B}}T$.

Only transitions between states corresponding to $\Delta J = \pm 1$ are allowed. So the $\Delta J = 0$ transition is missing. In **Figs. 3 and 4** the set of transitions in which the upper level has the higher J is called the **R** branch, while the **P** branch has this reversed. The transitions within a branch are labeled by the J of the lower level; **R(0)**, **P(1)** would be the labeling for the lowest J transitions.

(f) Draw a labelled energy level diagram, similar to **Fig. 4**, showing the first five rotational levels in the ground and first vibrational states ($n = 0 \rightarrow 1$). Show, using arrows as in **Fig. 4**, all of the transitions allowed between these states (and only these transitions), and label each transitions with the **P**, **R** notation.

(g) An estimate of the $J = 0$ transition energy can be obtained from the average of the **R(0)** and **P(1)** transition energies. If this corresponds to $8.66 \times 10^{13}\,\mathrm{Hz}$, using your result from **(d)**, calculate the value of the spring constant k for the HCl bond.

Data: Atomic mass of hydrogen is $1.0078\,\mathrm{u}$
Atomic mass of chlorine-35 is $34.9688\,\mathrm{u}$

Qu 3. Diffraction

Write you answer starting on a new page for this question.

Huygens's wave theory states that each point on a wavefront can be treated as a secondary point source. A wavelet (a circular ripple) spreads out from each point and the new wavefront is formed by drawing a smooth line through all the fronts of these wavelets (an envelope enclosing all the wavelets). i.e. the superposition of all the smaller diverging wavelets emitted by the oscillating particles will form a new wavefront. This is illustrated in **Fig. 5** below.

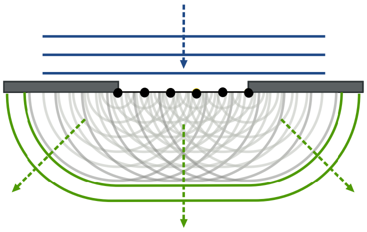

Figure 5: A set of points on a wavefront acting as sources for secondary wavelets with an envelope constructed to form a new wavefront.
Credit By Arne Nordmann (norro) - Own illustration, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=1944668

(a) Consider light diffracting through a single slit of width s_0 which is illuminated by a plane wavefront of uniform intensity and at normal incidence. We can determine the positions of the minima in a single slit pattern incident on a distant screen by considering the plane wavefront entering the slit to be constructed of a very large number of secondary points sources, each emitting circular wavelets (of light). In **Fig 6** are shown two "emitted waves".

(i) Comment on the relation between these two rays and the secondary sources in the slit emitting wavelets.

(ii) At a certain angle, the path difference between rays emitted from the centre of the slit and the edge of the slit is $\lambda/2$. Explain why the intensity of light falling on the screen in this direction is zero.

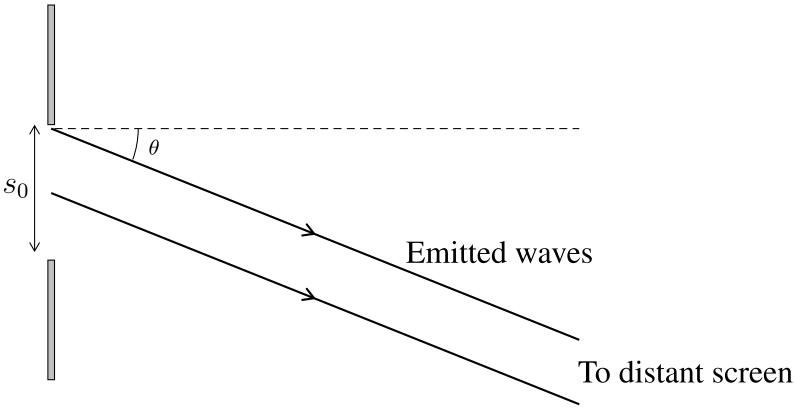

Figure 6: The path of two rays in the diffraction of light through a single slit.

(iii) By splitting the slit into $1, 2, 3, \ldots$ parts and pairing up the rays, deduce all angles, θ, for which the intensity of light falling on the screen is zero.

(iv) For a screen a distance D away, where $D \gg s_0 \gg \lambda$, deduce the fringe spacing of the diffraction pattern and the width of the central maximum.

(b) Finding the intensity of light on the screen as a function of angle θ.

(i) Explain why the total (superposed) wave emitted by the slit at any angle will oscillate in phase with the wave travelling from the point at the very centre of the slit.

You might consider the secondary sources in the slit emitting waves which have the form of $A\cos(\omega t + \phi(\delta s))$ where $\phi(\delta s)$ represents the phase of a wave emitted from a point a distance δs from the centre of the slit and in the plane of the slit.

(ii) Consider a wave incident on the slit with an amplitude dependent upon δs, taking the form $A(\delta s) = \frac{A_0}{s_0}\delta s$, where A_0 is the amplitude at the centre.

i. Sketch a graph of the amplitude of the wavefront from one side of the slit to the other.

ii. Sketch a graph of the intensity of the wavefront from one side of the slit to the other.

iii. Now taking a fixed amplitude A_0 across the slit, independent of δs, Show that the intensity of radiation as a function of θ, where θ is the angle measured from the centre line normal to the slit, is given by

$$I(\theta) = I(0)\frac{\sin^2\left(\frac{\pi s_0}{\lambda}\sin\theta\right)}{\left(\frac{\pi s_0}{\lambda}\sin\theta\right)^2} \tag{1}$$

Hint: Choose a point in time when the 'height' of the wave emitted from the centre of the slit is maximised.

The limit of $\frac{\sin^2 x}{x^2}$ *for* $x = 0$ *is 1.*

(c) Numerical results for $I(\theta)$.

(i) Light from a red laser of wavelength $628\,\mathrm{nm}$ is normally incident on a single slit of width $20\,\mu\mathrm{m}$ after which the light strikes a screen $1.00\,\mathrm{m}$ away. Find the distance, on the screen, from the centre of the diffraction pattern to the brightest point of the first order maximum.

Hint: The brightest spots are not directly in the middle of the two adjacent minima. You may assume that the first non-zero positive solution to the equation $x = \tan x$ *is, to four significant figures:* $x = 4.4934$

(ii) Find the ratio between the zeroth and first order maxima. At which order maximum does this ratio fall below 1%?

Hint: Assume that for $n \geq 2$ *the brightest point of a maxima falls roughly in the middle of the adjacent minima.*

Qu 4. Free Electron Model of Conduction in Metals

Write you answer starting on a new page for this question.

This question is about conduction in a metal. **Data on the metal sodium can be found at the end of the question.** (Sodium has one free electron per atom for conduction.)
Conduction in metals can be usefully discussed in terms of the free electron model. We can consider "free" electrons that move with complete freedom throughout a metallic solid, subject only to reflection at the inside surface of the conducting body. Classically the electron has a large random thermal motion with a superimposed systematic drift due to an electric or thermal gradient in the metal.

(a) If we consider the electron gas to have a classical gas distribution of velocities, using the kinetic interpretation of temperature, evaluate the RMS velocity of an electron at room temperature ($300\,\mathrm{K}$). This is equivalent to the thermal speed, v_{th}.

(b) If an electric field E is applied to the conductor the electrons will drift in the opposite direction to E. This drift velocity v_{d} is lost due to random collisions of the electrons with the ion lattice, so that the average drift velocity is zero immediately after any scattering collision.

- **(i)** Sketch a schematic graph of v_{d} against t for an electron in the field E.
- **(ii)** Calculate the average drift velocity of an electron in a sodium wire of cross sectional area $1\,\mathrm{mm}^2$ with a current of 1 A flowing.
- **(iii)** Sketch a path of such a conduction electron as it travels along the wire
- **(iv)** Calculate a value for the distance travelled parallel to the field, x_{d} in time τ_m.

(c) The resistivity of a metal, ρ is also often cited in terms of the electron mean free path, the distance $\lambda = v_{\mathrm{th}}\tau_m$ that any electron moves by virtue of its thermal speed during the mean free time between collisions with the lattice ions, τ_m. Derive an expression for the resistivity of a metal, in terms of $e, \lambda, n, k_{\mathrm{B}}, m_e, T$.

(d) The temperature dependence of the resistivity of a metal is obtained in this result. How does this dependence compare with the typical temperature dependence of ρ for metals at room temperature?

(e) Using the conductivity $\sigma = 1/\rho$ for sodium, and the data given at the end of the question, calculate the mean free path of an electron in sodium at room temperature.

It is clear that this model of conduction gives the wrong temperature dependence, and including a distribution of electron velocities as for a gas does not correct the difficulty.
To glimpse an approach to a quantum model, we treat the electron as a standing wave with a de Broglie wavelength in a closed container (analogous to sound waves in a closed pipe). An electron is trapped in the metal conductor in a state represented by a standing wave with nodes at the faces. The states form energy levels in the solid (like energy levels in an atom), as illustrated in **Figure 7**. Due to a quantum mechanical effect (the Pauli Exclusion Principle) at most two electrons can occupy a single energy level: the next two electrons fill the next energy level, etc.

(f) If we take energy levels as determined by de Broglie waves with nodes at the surface of the metal, sketch the waves corresponding to the lowest three energy levels for six electrons trapped in a one dimensional potential well of width ℓ.

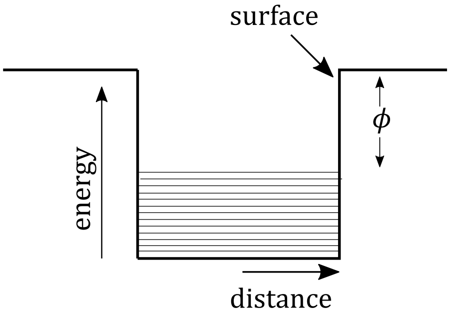

Figure 7

(g) In a 3-D cube of metal of side ℓ, the energy levels are filled up to an energy E_{F} by N electrons. Show that the number of energy levels filled up by N electrons is given by $E_{\mathrm{F}} = \frac{h^2}{8\ell^2 m_e}\left(n_x^2 + n_y^2 + n_z^2\right)$, where n_x, n_y, n_z are the labels of levels below filled by electrons.

If we let $(n_x^2 + n_y^2 + n_z^2) = n_{max}^2$ then all of the levels are filled up to n_{max}, which corresponds to a radius in an octant of a sphere in a space of radius n_{max} (the n's are only positive values), as shown in **Figure 8**. Thus at temperature $T = 0\,\mathrm{K}$ all N conduction electrons in a cube of side ℓ occupy L levels, given by

$$L = \frac{1}{8}\left(\frac{4\pi n_{max}^3}{3}\right)$$

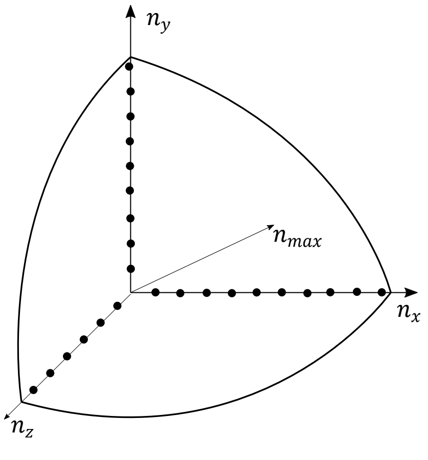

Figure 8

(h) Derive an expression for E_{F} in terms of h, m_{e}, N and volume $V = \ell^3$.

(i) Hence obtain a numerical value for E_{F} for sodium in units of eV.

At a temperature $T > 0\,\mathrm{K}$ only the states near E_{F} can supply conduction electrons, electrons in the lower states being fixed in their energy levels by the electrons above. (There are no empty states for these lower electrons to scatter into). This leads to a disconnect between the temperature and the classical thermal energy of the electrons.

(j) The Fermi speed, v_{F} is defined by $\frac{1}{2}mv_{\mathrm{F}}^2 = E_{\mathrm{F}}$, and the mean free path is now given by $\lambda = v_{\mathrm{F}}\tau_m$.

  (i) Calculate the value of λ for these scattered conduction electrons at the the Fermi energy.

(ii) How many atomic sodium diameters does this length correspond to? (assume a simple cubic packing structure for sodium to calculate a diameter).

(k) Calculate the derivative $\left(\dfrac{\mathrm{d}n}{\mathrm{d}E}\right)$ at E_{F}. Hence calculate the energy separation of two states at $E = E_{\mathrm{F}}$. It may be concluded that we may consider the energy to be virtually continuous for these conduction electrons.

Data for sodium at room temperature:

Sodium has one free electron per atom for conduction.

conductivity	$\sigma = 8.7 \times 10^{7}\,\mathrm{m}\,\Omega^{-1}$
electron ion number density	$n = 2.9 \times 10^{28}\,\mathrm{m}^{-3}$
density	$\rho_m = 0.971 \times 10^{3}\,\mathrm{kg}\,\mathrm{m}^{-3}$
molar mass	$M = 23.0\,\mathrm{g}\,\mathrm{mol}^{-1}$

END OF PAPER

Questions proposed by:
Dr James Bedford (Harrow School)
Dr Benjamin Dive (Austrian Academy of Sciences)
Robin Hughes (British Physics Olympiad & Isaacphysics.org)
Josh Brown (Trinity College, Cambridge)

BPhO Sponsors

NPL
National Physical Laboratory

Rolls-Royce

UNIVERSITY OF OXFORD

UNIVERSITY OF CAMBRIDGE

Maplesoft
Mathematics • Modeling • Simulation

TRINITY COLLEGE
CAMBRIDGE

SINE NOBIS SCIENTIA LANGUET

Worshipful Company of Scientific Instrument Makers

ROYAL ASTRONOMICAL SOCIETY
QUICQUID NITET NOTANDUM
1820

ONEWORLD

OXFORD
UNIVERSITY PRESS

PRINCETON
UNIVERSITY
PRESS

Springer

World Scientific
Connecting Great Minds

THE
RANDOM
HOUSE GROUP