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British Physics Olympiad 2022-23

Round 2 Competition Paper

February 2023

Instructions

Time: 3 hours (approximately: Q1 40 min, Q2 45 min, Q3 45 min, Q4 50 min).

Questions: All four questions should be attempted.

Marks: The questions carry marks indicated by the times above.

Solutions: Answers and calculations are to be written on loose paper or in examination booklets, and graph paper should be provided. Students should ensure their name and school is clearly written on all answer sheets and pages are numbered. A standard formula booklet with standard physical constants should be supplied.

Instructions: To accommodate students sitting the paper at different times, please do not discuss any aspect of the paper on the internet until March. **This paper must not be taken out of the exam room. All notes must be collected in**.

Calculators: Any standard calculator may be used, but calculators must not have symbolic algebra capability. If they are programmable, then they must be cleared or used in "exam mode".

Clarity: Solutions must be written legibly, in black pen (the papers are photocopied), and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam paper.

Important Constants

Constant	Symbol	Value
Speed of light in free space	c	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Elementary charge	e	$1.60 \times 10^{-19}\,\mathrm{C}$
Acceleration of free fall at Earth's surface	g	$9.81\,\mathrm{m\,s^{-2}}$
Permittivity of free space	ε_0	$8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Permeability of free space	μ_0	$4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
Mass of an electron	m_{e}	$9.11 \times 10^{-31}\,\mathrm{kg}$
Mass of a neutron	m_{n}	$1.67 \times 10^{-27}\,\mathrm{kg}$
Mass of a proton	m_{p}	$1.67 \times 10^{-27}\,\mathrm{kg}$
Radius of a nucleon	r_0	$1.2 \times 10^{-15}\,\mathrm{m}$
Planck's constant	h	$6.63 \times 10^{-34}\,\mathrm{J\,s}$
Gravitational constant	G	$6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Boltzmann constant	k	$1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Molar gas constant	R	$8.31\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Specific heat capacity of water	c_{w}	$4.19 \times 10^{3}\,\mathrm{J\,kg^{-1}\,K^{-1}}$
Mass of the Sun	M_{S}	$1.99 \times 10^{30}\,\mathrm{kg}$
Mass of the Earth	M_{E}	$5.97 \times 10^{24}\,\mathrm{kg}$
Radius of the Earth	R_{E}	$6.38 \times 10^{6}\,\mathrm{m}$

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Qu 1. General Questions

(a). Air bubbles in water are observed to rise towards the surface at different speeds according to their initial size. Use your knowledge of physics to suggest an explanation.

(b). Walking out of the front door on a cold, frosty morning, two effects are observed outside.

(i) A nail, hammered in to the wooden gate, has a splay of ice crystals growing from it, shown in **Fig. 1**. With a few brief statements, suggest a possible explanation of this effect.

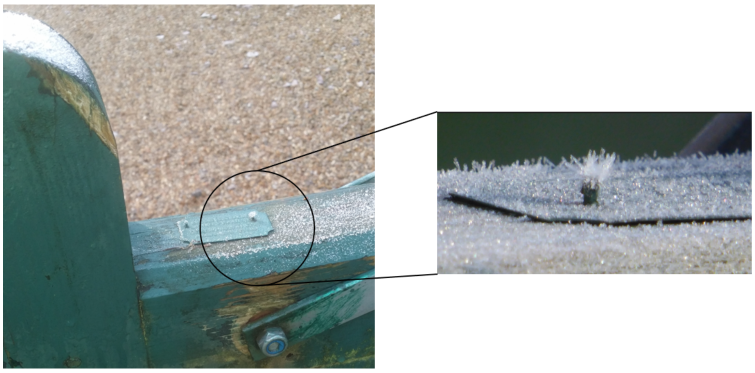


Figure 1: Ice crystals formed on a nail in a wooden fence on a frosty morning.

(ii) The windscreen of a car has a pattern on it, as in **Fig. 2**, which is not scratches on the glass, but ice. Again, with a few brief statements, suggest a possible explanation of this effect.

Figure 2: Ice crystals forming a pattern on the windscreen of a car on a cold, dry morning.

(c). Estimate the number of peas (the small, green edible vegetable) that are consumed in the UK in a year.

(d). A voltage amplifier of the type known as an op-amp consists of an integrated circuit constructed from many elements (transistors, resistors, etc.). Its internal workings are unimportant in this question. It is symbolised by a triangle with two inputs and an output, as shown in **Fig. 3**. It has a dc power supply to enable it to work, but that is not shown. The signal input terminals are labelled $+$ and $-$, which is a notation and not the polarity of voltage supplied. They are respectively the

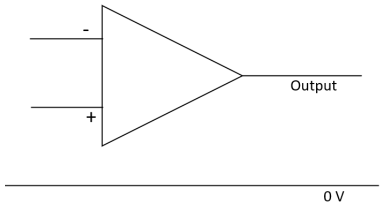


Figure 3: op-amp.

non-inverting input and the *inverting* input. We will assume a perfectly adjusted op-amp in order to make the following simplified statements.

- If the two inputs are connected together, the output is 0 V.
- If the inverting input (-) is set to 0 V then a small voltage, V_{in}, applied to the non-inverting input (+) will produce an output voltage $A_0 V_{\text{in}}$ where A_0 is the open-loop gain and $A_0 \gg 1$ (A_0 is typically of order $10^5 - 10^6$). If V_{in} is not small, then the output will saturate at the supply voltage and not at $A_0 V_{\text{in}}$.
- If the non-inverting input (+) is set to 0 V, and a small voltage V_{in} is applied to the inverting input (-) then the output voltage will be $-A_0 V_{\text{in}}$.

Three examples are shown in **Fig. 4**. The op-amp is rarely used in this simple manner as the A_0 is

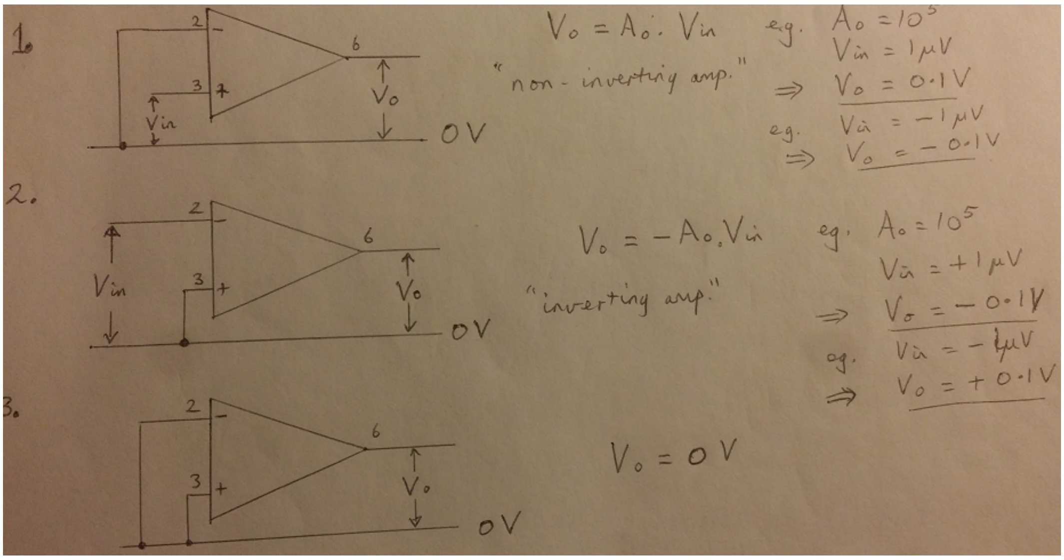


Figure 4: Three op-amp examples of the input and output values.

large but not known precisely. So usually, a small fraction of the output signal is fed back to the input and, as we shall see, this controls the gain of A_g of the circuit (A_0 is for the op-amp alone).

(i) In the circuit shown in **Fig. 5**, we can assume that negligible current enters the inverting terminal.

   i. Copy the circuit and sketch the path of the current i that flows into the circuit at V_{in} through the pair of resistors to V_{out}.

   ii. Given that the non-inverting input is at 0 V, and that the (small) potential at the inverting input is given by $v = -\dfrac{V_{\text{out}}}{A_0}$, obtain equations for the current i flowing through R_{in} in terms of V_{out}, V_{in} and A_0, and the same current i through R_{f} in terms of these quantities.

iii. Eliminating the current i between these two equations, obtain an expression for $A_g = \dfrac{V_{\text{out}}}{V_{\text{in}}}$.

iv. For $A_0 \gg 1$, obtain an expression for A_g in terms of R_{f} and R_{in}.

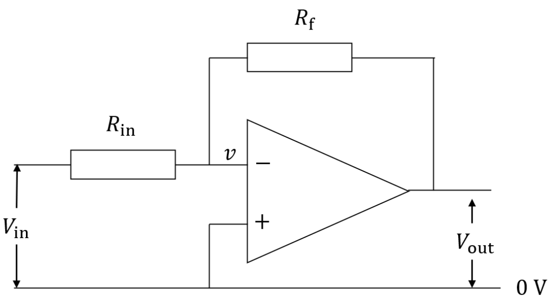


Figure 5: Inverting amplifier circuit with feedback from the output to the inverting input.

(ii) For the non-inverting amplifier of **Fig. 6** the non-inverting input is now at V_{in} whilst the inverting input is again at potential v given by $V_{\text{out}} = A_0(V_{\text{in}} - v)$. A current i flows from the output down to 0 V with no current entering the inverting input.

i. Write down equations for the current i flowing through R_1 and R_2 in terms of V_{out}, and the same current through R_2 in terms of v.

ii. Eliminating the current i between these two equations, obtain an expression for $A_g = \dfrac{V_{\text{out}}}{V_{\text{in}}}$.

iii. For $A_0 \gg 1$, obtain an expression for A_g in terms of R_1 and R_2 only.

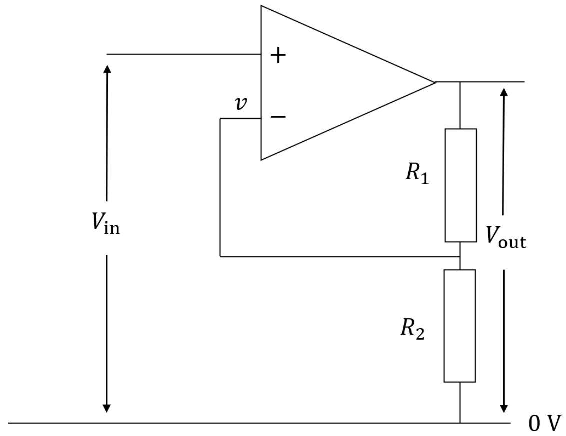


Figure 6: Non-inverting amplifier circuit with feedback from the output again to the inverting input.

(iii) In the following circuit, we assume that $A_0 \gg 1$ and so v is small enough that the two inputs can be taken to be the same value and $v \approx 0$, and also that no current enters the op-amp inputs (these are the two "Golden Rules" for op-amps).

For the circuit of **Fig. 7** a current i flows through R_{in} and onto the capacitor when switch S is opened at time $t = 0$. No current enters the inverting input.

i. Obtain equations for the current i in terms of R_{in} and V_{in}, and then the charge Q on the capacitor in terms of C and V_{out}.

ii. Obtain a differential equation $\dfrac{dV_{\text{out}}}{dt}$ which can then be expressed in terms of $V_{\text{in}}, R_{\text{in}}$ and C. Solve this for V_{out} in terms V_{in}.

iii. Sketch a graph of V_{out} against time, t.

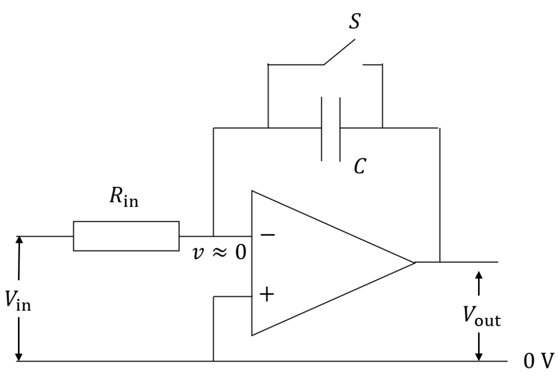


Figure 7: Circuit with feedback via a capacitor to the inverting input.

Qu 2. Energy Levels

This question explores energy levels in atoms. The model is interesting both from a historical perspective and as an example of where classical physics meets quantum physics.

(a). In a simple model of the hydrogen atom due to Bohr and de Broglie, electrons are assumed to follow circular orbits around the nucleus with the wave-like nature of electrons manifest as resonant states or standing waves around a closed orbit, with the number of half-wavelengths given by the integer, n. This model of the orbits is illustrated in **Fig. 8**.

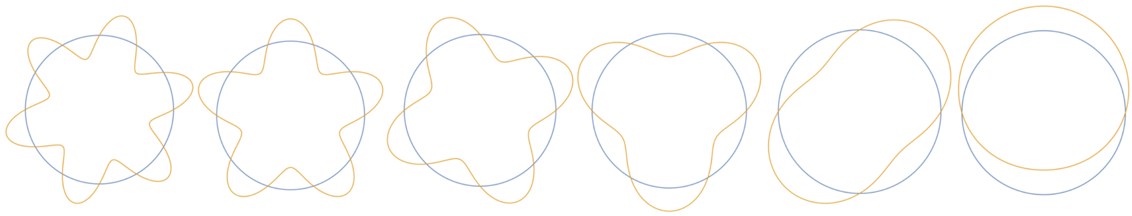


Figure 8: Bohr orbits, pictured as standing waves of an electron in a closed orbit around the nucleus.

(i) Sketch the $n = 3$ example shown and referring to your sketch, comment on what determines the value of n.

(ii) By considering the electron simply as a particle of mass m and charge e orbiting the nucleus, derive an expression for the total energy of the system, E, in terms of the radius of the orbit of the electron around the nucleus, r

The energy of the electron in its orbit is dependent upon r, but there is no restriction on the value of r or of E. Now, if we consider the electron as a de Broglie wave,

(iii) show that the wave-like nature illustrated in **Fig. 8** leads to the quantisation condition

$$rp = n\frac{h}{2\pi}$$

where p is the momentum of the electron.

(iv) Determine the radius of the hydrogen atom that this predicts when $n = 1$. Does it seem sensible?

(v) Determine the energy levels of atomic hydrogen, E_n, in this model. Calculate the ground state energy in eV.

(b). In a slightly improved model, the electron and nucleus orbit their common centre of mass.

(i) Show that the electron and nucleus orbit with a common angular velocity, ω, and find an expression for ω in terms of r, the separation of the electron from the nucleus, and μ, the so-called 'reduced mass'

$$\mu = \frac{mM}{m + M}$$

with m the mass of the electron and M the mass of the nucleus.

(ii) The quantity rp in **(a).(iii)** is the angular momentum of the electron. This is the 'moment of momentum' about the orbital centre. A generalisation of the quantisation condition arrived at previously has the total (sum of) angular momentum of the electron and nucleus quantised in units of $\frac{h}{2\pi}$. Calculate the lowest two energy levels of both atomic hydrogen and atomic tritium in this model.

(iii) Which of hydrogen and tritium would absorb a shorter wavelength of light for a given atomic transition? What is the difference between the two wavelengths?

Qu 3. Party balloons

(a). Explain why a heavy balloon inflated with helium-4 gives a lower reading on an accurate weighing scale than the same balloon when deflated, despite having a larger mass.

A party balloon of mass $10\,\mathrm{g}$ is inflated with helium-4.

(b). Show that, under typical atmospheric conditions, the volume of the balloon V should satisfy $V \gtrsim 1.0 \times 10^{-2}\,\mathrm{m}^3$ if it is to float.

The balloon is initially inflated to a volume $V_0 = 2.0 \times 10^{-2}\,\mathrm{m}^3$, then released. It floats up gently before coming to rest on the ceiling. Unfortunately, there is a leak, and the balloon slowly deflates through a very small hole of area A.

(c). Show that the root mean-square momentum P_{rms} of a particle in a pure ideal gas of temperature T is

$$P_{\text{rms}} = \sqrt{3mkT},$$

where m is the particle mass. Hence reason why the typical momentum in any given direction is

$$P = \sqrt{mkT}.$$

(d). The pressure of a gas can be loosely thought of as the total momentum flowing through unit area per unit time. Making the approximation that all particles travelling in a given direction have momentum P, explain why N, the number of atoms of helium-4 in the balloon, decreases over time as:

$$\frac{dN}{dt} = -\frac{pA}{\sqrt{mkT}}.$$

(e). Hence obtain an expression for the rate of change of the volume, $\frac{dV}{dt}$, in terms of A, k, T, m.

(f). If after 1 hour the balloon has shrunk by 5% in volume, estimate the time until the balloon drops back to the floor, and estimate the area of the leaky hole. How does this compare to the initial surface area of the balloon?

(g). Suggest why the model presented above would break down if the hole was made significantly larger.

(h). The balloon was itself filled from a rigid, pressurised cylinder of helium-4 of fixed volume v. If this cylinder also develops a small leak (of area a), derive an algebraic expression for the gas pressure inside the cylinder as a function of time. Take the initial pressure to be p_0. No numerical calculations are needed.

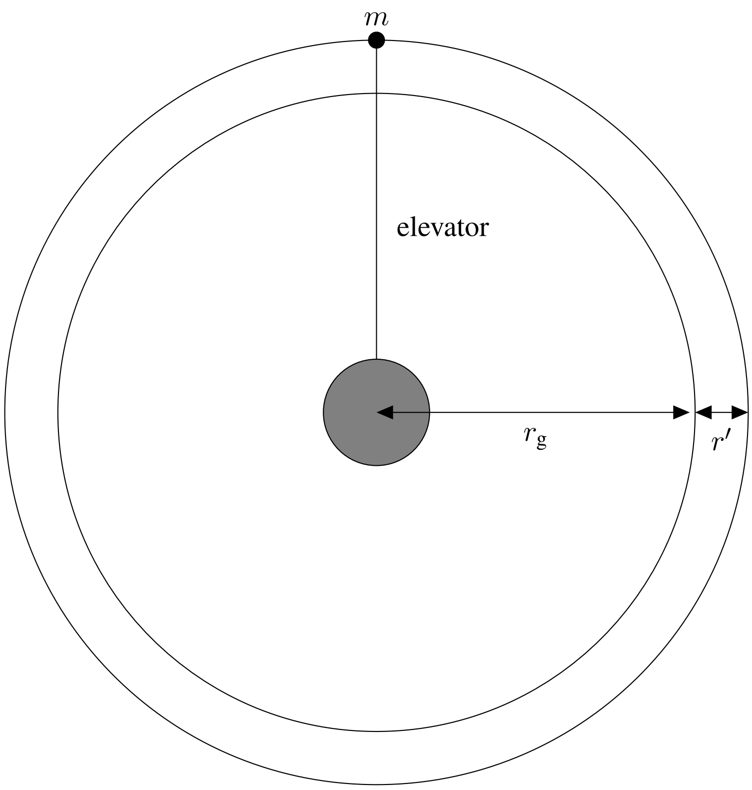
Qu 4. A Space Elevator

This question explores the idea of a space elevator for reaching a high altitude orbit such as a geostationary one.

When considering the physics of rotating objects, a helpful way to analyse the motion can be to transform to a frame of reference which is rotating with the object, and in which the object is therefore (in the simplest case) at rest. When an object of mass m is at rest in this frame of reference, and the reference frame is rotating uniformly with angular velocity ω, the transformation to this frame gives rise to a fictitious 'centrifugal' force equal to $mr\omega^2$, where r is the distance of the mass from the axis of rotation. The force is centrifugal in that it is directed away from the axis of rotation rather than towards it. To handle the laws of motion in such a rotating frame, simply treat the fictitious forces like real forces and go about things as usual!

(a). Find an expression for the orbital time period for a satellite of mass m_{s} orbiting at a distance r_{g} from the centre of the earth. Calculate the radius of the orbit of a geostationary satellite.

(b). A mass m is to be kept in orbit at a radius $r_{\mathrm{g}} + r'$ ($r' \ll r_{\mathrm{g}}$) as shown in the diagram below. What extra force is required to do this over and above the gravitational force. Determine both its magnitude and direction.



(c). An elevator is hung from the mass m as shown in the diagram above and reaches all the way down to the surface of the earth. Modelling the elevator as a thin rod with mass per unit length μ, show that for the elevator plus mass to maintain the same geostationary orbital period then the mass must (taking the radius of a geostationary orbit to be very much bigger than the radius of the earth) be given by

$$m = \frac{\mu r_{\mathrm{g}}^3}{3 r_{\mathrm{e}} r'}$$

where r_{e} is the radius of the earth.

(d). Using reasonable estimates for μ and r', comment on the viability of this proposal using standard materials.

(e). Contemplate instead a 'free-standing' elevator without a counterweight, modelled as a uniform cable whose ends require no restraint. Again the cable rotates with the earth. Consider the net stress $\mathrm{d}\sigma$ on an element of cable $\mathrm{d}r$ with cross-sectional area A at a distance r from the centre of the earth. Show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}r} = Gm_{\mathrm{e}}\rho\left(\frac{1}{r^2} - \frac{r}{r_{\mathrm{g}}^3}\right)$$

where ρ is the mass density (by volume) of the cable and m_{e} is the mass of the earth.

(f). What is the largest tensile stress in the cable and at what radius does it occur? Compare this for the three different materials in the table below and comment on your findings.

Material	**Density** $/\mathrm{kg\,m^{-3}}$	**Maximum tensile stress*** $/\mathrm{GPa}$
Steel	7900	5.0
Kevlar	1400	3.6
Carbon nanotubes	1300	130

* Safe practice typically dictates that materials are not subject to stresses of more than half their maximum.

(g). A different design of elevator is one where the stress is held fixed (at a safe value) and instead the cross-sectional area varies. Show that in this setup the cross-sectional area is given by

$$A(r) = A_{\mathrm{b}} \exp\left[\frac{r_{\mathrm{e}}^2}{L_{\mathrm{c}}}\left(-\frac{1}{r} - \frac{r^2}{2r_{\mathrm{g}}^3} + k\right)\right]$$

where k is a constant to be determined, A_{b} is the area at the base of the elevator and $L_c = \frac{\sigma}{\rho g}$ (with g the acceleration due to gravity at the surface of the earth) is the 'characteristic length' of the elevator material. What is the maximum area and at what radius does it occur?

(h). For a realistic construction, the ratio between the area at ground level and the maximum area should not be too large. Determine this ratio (known as the 'taper ratio') for the materials given above (for a suitable stress) and comment on your findings.

(i). Assuming a symmetrical construction of the variable area elevator, calculate its height and compare the value you obtain with the height of the setup in **(e).**

END OF PAPER

Questions proposed by:
Rupert Allison (Harrow School)
James Bedford (Harrow School)
Robin Hughes (Isaac Physics & BPhO)