United States Physics Team

Entia non multiplicanda sunt praeter necessitatem

1997 Exam 2

Instructions:

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

1. **Please show all work, and do not hesitate to use explanatory words in addition to equations. Partial credit will be awarded**.

2. Begin each problem on a fresh sheet of paper. In the upper right-hand corner of each page, write your name, problem number, and the page number/total number of pages for this problem (e.g., "A2, 3/4" for "Problem A2, page 3 of four").

3. You may use a non-programmable calculator, but may not use any tables, books, notes, or collections of formulas.

4. Work Part A first. You have ninety minutes to complete all four problems.

5. After you have completed Part A, you may take a break.

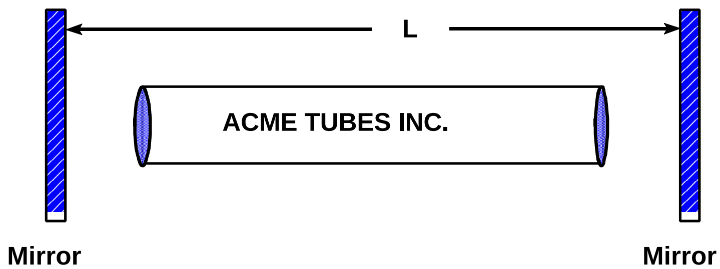
6. Then work Part B. You have 90 minutes to complete both problems.

Possibly useful information:

Gravitational field on Earth's surface: $g = 9.8$ N/kg
Gravitational constant: $G = 6.67\times10^{-11}$ N-m^2 / kg^2
Coulomb constant: $k = 1 / 4\pi\epsilon_o = 8.99\times10^9$ N-m^2 / C^2
Speed of light in vacuum: $c = 3.00\times10^8$ m/s
Permeability of empty space: μ_o , where $\mu_o\epsilon_o = 1/c^2$
Ideal gas constant $R = 8.31$ J / mol-K
Boltzmann's constant: $k_B = 1.38\times10^{-23}$ J/K
Avagadro's number: $N_A = 6.02\times10^{23}$
1 electron volt = 1 eV = 1.60×10^{-19} J
Planck's constant: $h = 6.63\times10^{-34}$ J-s = 4.14×10^{-15} eV-s

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$ for $|x| < 1$
$(1 + x)^n = 1 + nx + (1/2!)n(n-1)x^2 + \ldots$ for $|x| < 1$
$\cos\theta \approx 1 - \theta^2/2$ for $\theta << 1$
$\sin\theta \approx \theta$ for $\theta << 1$

A1. A laser may be constructed from two main components: a resonant cavity of length L formed from two highly reflecting flat mirrors, and a gas-filled tube that emits light when excited by a high voltage. Placed inside the otherwise evacuated cavity, the tube creates light which bounces back and forth between the mirrors such that standing light waves are produced within the cavity. (The radiation can be let out through a tiny window in one of the mirrors, but in this problem we are interested in what goes inside the cavity.)



(a, 5) What are the frequencies possible for the standing light waves in the cavity? Express your answer in terms of the cavity length L and the speed of light in vacuum, c.

(b, 5) Suppose the plasma tube emits light of a pure frequency $f_o = 5\times10^{14}$ Hz. Which standing wave mode (specified by an integer n_o) is excited in the cavity if $L = 1.5$ m?

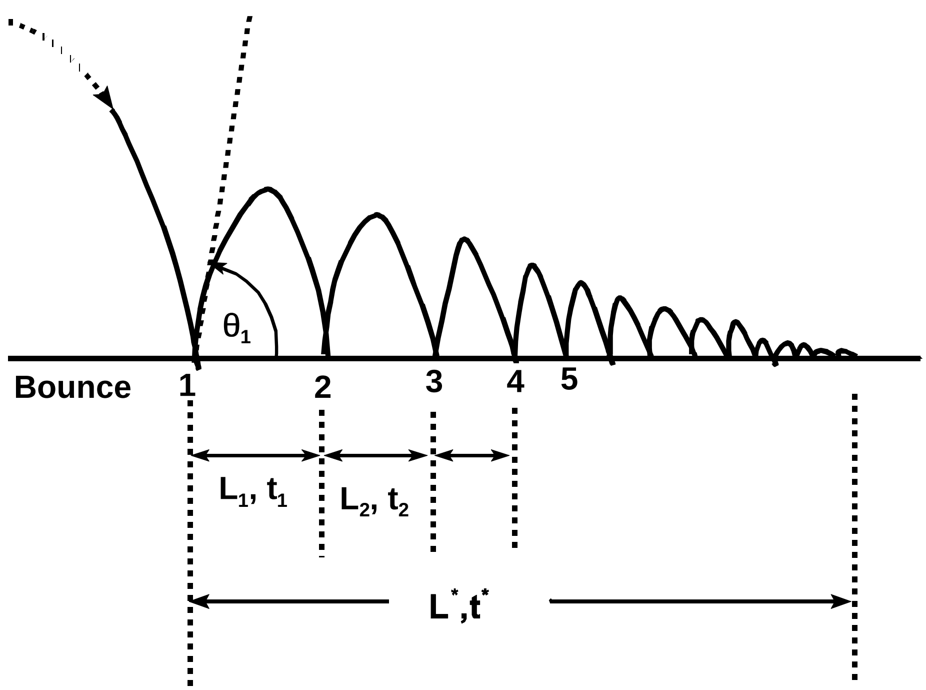
(c, 5) Suppose the plasma tube emits light not of a pure frequency, but emits light having all the frequencies in the range $f_o \pm \Delta f$, where $f_o = 5\times10^{14}$ Hz, and $\Delta f = 1\times10^{9}$ Hz. How many standing wave modes are excited in the cavity if $L = 1.5$ m?

(d, 5) Using the f_o and Δf of part (c), what is the largest value of L such that only *one* standing wave mode will be excited in the cavity, thereby giving the laser only *one* output frequency?

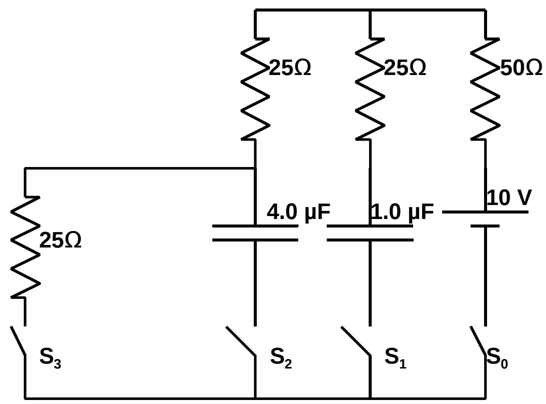
(Adapted from A.P. French, *Vibrations and Waves*, Norton, NY, 1971).

A2. **(20)** A ball is tossed onto the floor, where it makes a succession of bounces as illustrated in the figure below. Assume that because of internal elasticity and friction with the floor, at each bounce the magnitude of the vertical velocity component is reduced by a factor ϵ_y and the horizontal component is reduced by a factor ϵ_x. That is, if $v_{oy,n+1}$ denotes the y-component of the velocity as the ball emerges from the $(n+1)$st bounce, then $v_{oy,n+1} = \epsilon_y v_{oy,n}$, and similarly for the x-component. Note that ϵ_y and ϵ_x are < 1. Thus after each bounce, the ball moves slower and hops a shorter distance than it did after the preceding bounce.

Let the ball's succession of bounces traverse a total horizontal distance L^* which takes the time t^*. (As a practical matter, we measure L^* and t^* as the length and time where the bounces become imperceptible; mathematically, the number of bounces goes to infinity.) Find θ_1, the angle the ball's velocity makes with the horizontal immediately after the first bounce, written in terms of L^*, t^*, ϵ_y, ϵ_x, and needed constants. Neglect air resistance.



A3. The circuit shown consists of:



a 10 volt battery,
a 1.0 μF capacitor,
a 4.0 μF capacitor,
a 50 Ω resistor,
three 25 Ω resistors, and four switches, labeled S_o, S_1, S_2, S_3 . All switches are initially open. In all cases assume that the circuit is completely insulated from its surroundings.

(a, 5) Switches S_o, S_1, S_2 are closed. Switch S_3 remains open. After a very long time, what is the charge on each capacitor?

(b, 5) Switch S_3 is also closed, so that all four switches are closed. After a very long time, what is the charge on each capacitor?

(c, 5) Switches S_o and S_3 are opened simultaneously. Switches S_1 and S_2 are left closed. After a very long time, what is the charge on each capacitor?

(d, 5) Switches S_2 and S_3 are opened. Switches S_o and S_1 are closed. After a very long time has passed you insert into the 1.0 μF capacitor a slab of dielectric material. After the dielectric is inserted, it totally fills the region between the plates. The material's dielectric constant is 3.0. What is the total work done in inserting the dielectric (that it, what is the net work done by you *and* the battery)?

A4. An atom in an excited state typically decays by emitting a photon. The atom is originally in some excited state S', and after the decay it is in the ground state S. We usually think of the photon as carrying away the energy difference between states S' and S. In this problem we examine this process closely.

(a, 4) Consider a stationary, isolated atom in some excited state S'. Before decay the mass of the atom is M'. The atom decays into the ground state S. In the ground state the mass of the atom is M. Thus the energy $(M'-M)c^2$, which we denote here as E_o , is the energy that is released by the atom, and is therefore an upper limit on the energy that the photon could possibly carry away (c is the speed of light). Let E denote the energy of the outgoing photon. Without solving any equation, by merely giving an qualitative argument, explain why E must be strictly less than E_o in the case of an isolated excited atom undergoing radiative decay.

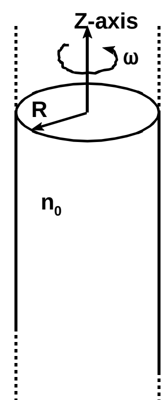
(b, 6) Quantify your answer to part (a) by showing that, approximately,

$$E \approx E_o [\, 1 - (E_o / 2Mc^2)\,], \text{ where } E_o / 2Mc^2 << 1.$$

(c, 5) Consider an atom decaying from the excited state S' to the ground state S, where $Mc^2 = 2\times10^{11}$ eV and $E_o = 4\times10^5$ eV. A photon energy corresponds to a radiation frequency. Thus the energy difference $E_o - E$ corresponds to a frequency shift $\Delta f = f_o - f$. Calculate the fractional frequency shift $\Delta f / f_o$.

(d, 5) Another way to produce a frequency shift is the Doppler effect, produced by relative motion between source and observer. The atom of part (c) emits the photon energy E, which corresponds to light frequency f. With what speed and in what direction would an observer have to move relative to this photon, to Doppler-shift its frequency back to the value f_o corresponding to the energy E_o ? The observer's speed v is $<< c$.

B1. Consider an infinitely long, cylindrical cloud of electrons. The cloud has radius R, the number density of electrons has the uniform value n_o , the charge of each electron is $-e$ (where $e > 0$), and the mass of each electron is m. The cloud is surrounded by vacuum. There is a constant, uniform magnetic field along the cylinder's axis (the z-axis), so that $\mathbf{B} = B_o\mathbf{k}$, with $B_o > 0$. The cloud of electrons is rotating about the z-axis with angular velocity ω as shown in the figure. The speeds are non-relativistic. [A gas of charged particles is called a "plasma."]



(a, 6) Find the electric field at the radius $r < R$ within the cloud.

(b, 6) Find the total force on an electron located at the radius r within the cloud, neglecting any magnetic field induced by the cloud's rotation. (This neglect will be justified in part (f) below.)

(c, 8) Using Newton's second law, show that there are two values of ω possible for the electrons to steadily orbit the cylinder's axis. Write these values of ω in terms of two other frequencies relevant to such problems: the cyclotron frequency $\omega_c = e B_o/m$, and the "plasma frequency" $\omega_p = [n_o e^2 / \epsilon_o m]^{1/2}$, where ϵ_o is the permittivity of empty space.

(d, 6) Find the largest electron number density $(n_o)_{max}$ that can be confined by the applied magnetic field $B_o\mathbf{k}$. Write your answer in terms of the magnetic field energy density $B^2/ 2\mu_o$ and the electron's mass energy mc^2.

[This theoretical limit is called the *Brillouin limit*, and plays a key role in the confinement of plasmas consisting of particles all of like charge.]

(e, 8) At a point located at radius r within the cylindrical cloud, calculate the magnetic field induced by the cloud's rotation. Assume that the field induced on the axis of symmetry is zero.

(f, 6) Determine the force on a charge in the cloud, due to the induced magnetic field of part (e). Show that the magnitude of this force divided by the magnitude of the electric force is v^2/c^2, where v is the speed of the charge and c is the speed of light. [This shows that the effect of the induced field is ignorable when the rotation speeds are non-relativistic.]

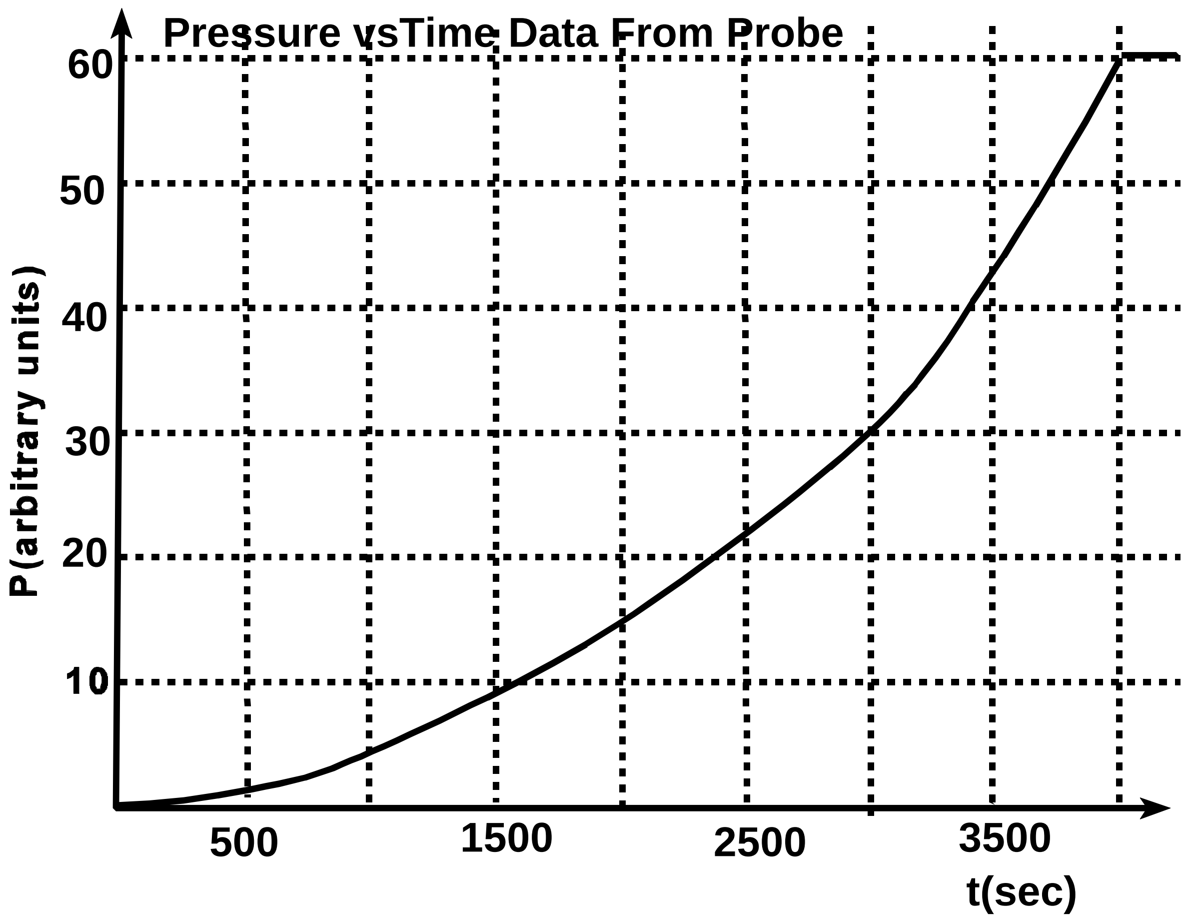
B2. An unmanned space probe approaches the surface of a planet whose atmosphere is pure CO_2 (molecular weight M = 44 gm/mol). Upon entering atmosphere, the probe descends straight down at a constant speed v_o , recording information about the atmospheric pressure, which is shown on the attached graph. Unfortunately, a technician forgot to calibrate the pressure-measuring instrument, so that the P-axis on the pressure vs. time graph has no units! (The technician has been reprimanded, but fortunately, as leader of the data analysis team, you have figured out how to use the data anyway.) Upon reaching the planet's surface, the probe reports the surface temperature to be 400 K, and the gravitational field there to be 9.9 N/kg. The radius of the planet is 5.0×10^6 m. Model the atmosphere locally as an ideal gas.

(a, 5) Apply Newton's Second Law to a small slab of the atmosphere of thickness Δy that is in static equilibrium, to show that the change in pressure ΔP between the top and bottom sides of the slab is given by $\Delta P = \pm\rho g\Delta y$, where ρ the atmosphere's density and g is the local gravitational field (the plus or minus sign is determined by the choice for the positive direction of the vertical y-axis; work out the sign in terms of your choice).

(b, 15) Estimate the velocity v_o with which the probe descends to the surface.

(c, 15) Estimate the temperature of the atmosphere 15 km above the surface. Neglect the variation of the planet's gravitational field between the surface and this height.

(d, 5) Justify the neglect of the variation of the planet's gravitational field between the surface and 15 km above the surface. Suggestion: Calculate the fractional difference between the gravitational field's magnitude on the surface and its value 15 km above the surface, and assess whether that difference is significant compared to the accuracy of the estimate made in part (c).



Pressure vsTime Data From Probe
60
50
40
30
20
10
P(arbitrary units)
500
1500
2500
3500
t(sec)