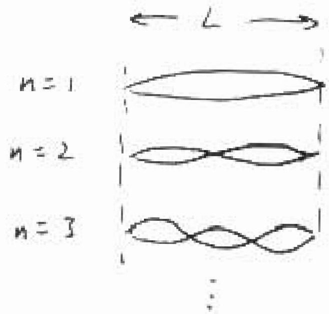
(a) The modes are



$$L = \frac{n\lambda}{2} \qquad \text{where } c = \lambda f$$

$$\Rightarrow \quad f_n = \frac{nc}{2L} \qquad n = 1, 2, 3, \ldots$$

(b)

$$n_0 = \frac{2L f_{n_0}}{c} = \frac{2(1.5\,\text{m})(5\times10^{14}/\text{s})}{3\times10^{8}\,\text{m/s}} = 5\times10^{6}$$

(c)

$$f_{n_0} \pm \Delta f = (n_0 \pm \Delta n)\frac{c}{2L}$$

$$\cancel{f_{n_0}} \pm \Delta f = \cancel{\frac{n_0 c}{2L}} \pm \frac{c}{2L}\Delta n$$

$$\Delta n = \frac{2L}{c}\Delta f = \frac{2(1.5\,\text{m})(1\times10^{9}/\text{s})}{3\times10^{8}\,\text{m/s}} = 10$$

So the frequencies being excited into resonance are those corresponding to $\underbrace{n_0,\ n_0 \pm 1,\ n_0 \pm 2,\ \ldots,\ n_0 \pm 10}_{21 \text{ modes}}$

(d) From (c), we note that as L gets shorter, Δn gets smaller (Δf is fixed by the plasma tube, not the cavity). Decreasing L, the first modes to go away are $n_0 \pm 10$, next is $n_0 \pm 9$, etc. To lop off ± 10 modes, the length L must be $\frac{1}{10}$ as long as before (since $L \propto \Delta n$). Hence

$$L = \frac{1.5\,\text{m}}{10} = 15\text{ cm}.$$

Each hop is a projectile motion, so that

$x = v_{ox}\, t$, (2)

and

$t = 2\, v_{oy} / g$ (from $0 = y = v_{oy}\, t - \frac{1}{2} g t^2$) . (2)

and upon eliminating t, the range L of a hop is

$L = 2\, v_{ox}\, v_{oy} / g$. (2)

Since $v_{oy,n+1} = \epsilon_y\, v_{oy,n}$ and $v_{ox,n+1} = \epsilon_x\, v_{ox,n}$, it follows that

$v_{oy,n+1} = \epsilon_y^{\,n}\, v_{oy,1}$ (1)

$v_{ox,n+1} = \epsilon_x^{\,n}\, v_{ox,1}$ (1)

$L_{n+1} = (\epsilon_y \epsilon_x)^n\, L_1$ (2)

$t_{n+1} = L_{n+1} / v_{ox,n+1} = \epsilon_y^{\,n}\, t_1$ (2)

The total range is

$$
\begin{aligned}
L^* &= L_1 + L_2 + L_3 + L_4 + \ldots \\
&= L_1 [\, 1 + \epsilon_y\epsilon_x + (\epsilon_y\epsilon_x)^2 + (\epsilon_y\epsilon_x)^3 + \ldots \,] \\
&= L_1 [\, 1 - \epsilon_y\epsilon_x \,]^{-1} \qquad (2)
\end{aligned}
$$

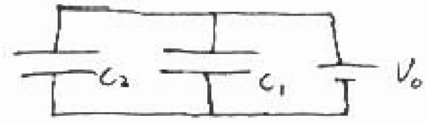
The total time is

$$
\begin{aligned}
t^* &= t_1 + t_2 + t_3 + t_4 + \ldots \\
&= t_1 [\, 1 + \epsilon_y + (\epsilon_y)^2 + (\epsilon_y)^3 + \ldots \,] \\
&= t_1 [\, 1 - \epsilon_y \,]^{-1} \qquad (2)
\end{aligned}
$$

The angle θ_1 is given by

$$
\begin{aligned}
\theta_1 &= \tan^{-1} (v_{oy,1} / v_{ox,1}) \\
&= \tan^{-1} (g\, t_1^2 / 2\, L_1) \\
&= \tan^{-1} \{ [g\, t^{*2} (1 - \epsilon_y)^2] / [\, 2\, L^* (1 - \epsilon_y\epsilon_x) \,] \} \qquad (4)
\end{aligned}
$$

(a) as $t \to \infty$ The circuit becomes:



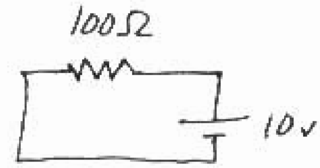
In parallel, so

$$V_1 = V_2 = V_0 = 10\,V$$

$$\Rightarrow \quad q_1 = C_1 V_0 = (1\mu F)(10\,V) = 10\,\mu C$$

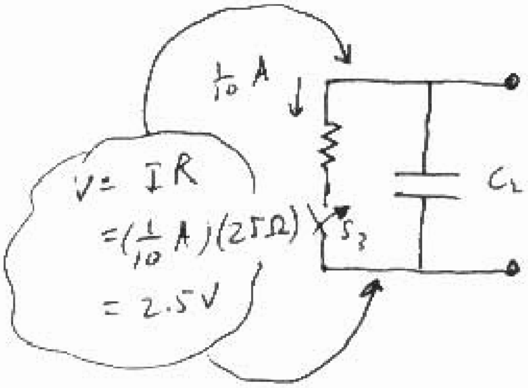
$$q_2 = C_2 V_0 = (4\mu F)(10\,V) = 40\,\mu C$$

(b) as $t \to \infty$, current flows through outer loop only,



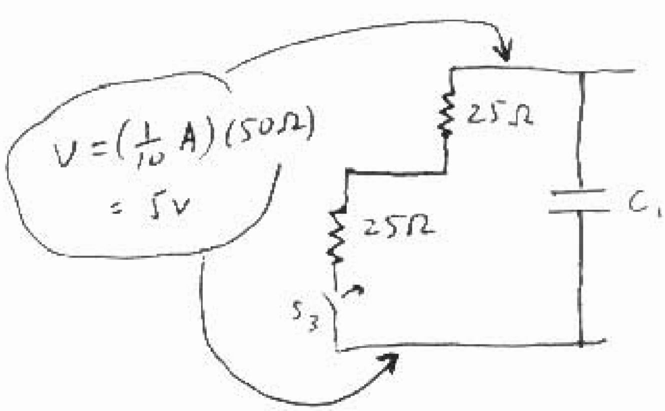
$$I = \frac{10\,V}{100\,\Omega} = \frac{1}{10}\,A$$

C_2 is in parallel with the 25Ω resistor by S_3:



$$\Rightarrow \quad q_2 = C_2 V_2 = (4\mu F)(2.5\,V) = 10\,\mu C$$

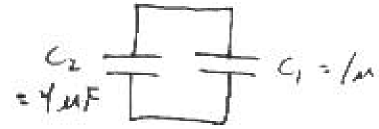
C_1 is in parallel with 50Ω



$$q_1 = C_1 V_1 = (1\mu F)(5\,V) = 5\,\mu C$$

(c) From (b), $q_{total} = q_1 + q_2 = 15\,\mu C$

Now the charge may only redistribute itself until $V_1 = V_2$

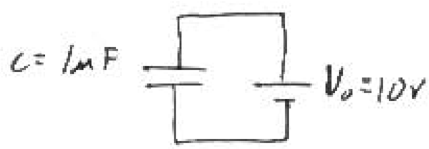


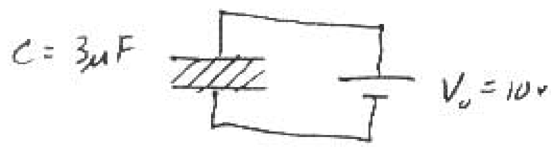
$q_1 + q_2 = 15\,\mu C$ and $\dfrac{q_1}{C_1} = \dfrac{q_2}{C_2}$

$q_1 = \dfrac{C_1}{C_2} q_2 = \frac{1}{4} q_2$

$\Rightarrow \frac{5}{4} q_2 = 15\,\mu C$ or $q_2 = 12\,\mu C$ $\Rightarrow$ $q_1 = 3\,\mu C$

(d) <u>initial</u> <u>final</u>





$q_i = (1\mu F)(10V) = 10\,\mu C$

$q_f = (3\mu F)(10V) = 30\,\mu C$

Battery connected, V does not change, but q does

$$\text{Work} = \Delta(\text{Energy})$$

$$W_{Battery} + W_{you} = \Delta\left(\tfrac{1}{2} C V^2\right)$$

$$= \tfrac{1}{2}(3C_1 - C_1)V^2$$

$$= \tfrac{1}{2}(2\mu F)(10V)^2$$

$$= 100\,\mu J$$

(a) To conserve momentum the atom must recoil. Thus after emitting the photon, the atom carries some kinetic energy K, leaving <u>less</u> than E_0 for photon.

(b) Before decay | After decay

M' | $p \leftarrow M$ photon $\cdot \longrightarrow$

(Energy, momentum) $= (M'c^2, 0)$ | $(Mc^2+K, -p)$ $(E, E/c)$

Conservation of Energy | Cons. of Momentum

$$M'c^2 = Mc^2 + K + E \qquad 0 = \frac{E}{c} - p$$

$$(M'-M)c^2 = K + E \qquad p = \frac{E}{c}$$

$$E_0 = K + E$$

since the atom is heavy, $v \ll c$ and $K = p^2/2m$, or $p \simeq \sqrt{2mK}$ for <u>atom</u>

given $\frac{E_0}{2mc^2} \ll 1$

since K and E are $< E_0$, $\frac{K}{2mc^2}$ and $\frac{E}{2mc^2}$ are $\ll 1$ too

$\therefore$

$$\sqrt{2mK} = \frac{E}{c} \Rightarrow K = \frac{E^2}{2mc^2}$$

or, relativistically,

$$(Mc^2)^2 = (K + Mc^2)^2 - (pc)^2$$

$$\Rightarrow K = Mc^2\left[\sqrt{1 + \left(\frac{E}{Mc^2}\right)^2} - 1\right]$$

$$\simeq E^2/2mc^2$$

Conservation of Energy Equation becomes

$$E_0 = \frac{E^2}{2mc^2} + E$$

$$\Rightarrow \quad E = E_0\left(1 + \frac{E}{2mc^2}\right)^{-1}$$

$$\simeq E_0\left(1 - \frac{E}{2mc^2}\right) \qquad \text{Iterate}$$

$$= E_0\left[1 - \frac{E_0(1 - E/2mc^2)}{2mc^2}\right]$$

Alternate: Quadratic formula

$$E^2 + \gamma E + \gamma E_0 = 0$$

where $\gamma = 2mc^2$

$$\Rightarrow E = \frac{\gamma}{2}\left[-1 \pm \left(1 + \frac{4E_0}{\gamma}\right)^{1/2}\right]$$

$$\simeq \frac{\gamma}{2}\left[-1 \pm \left(1 + 2\frac{E_0}{\gamma} - \frac{2E_0^2}{\gamma^2}\right)\right]$$

since $E > 0$ take $+$

$$E = E_0(1 - \gamma)$$

Note: 2nd order needed in Binomial

If $\delta \equiv \dfrac{E_0}{2mc^2} << 1$ Then $\dfrac{E E_0}{(2mc^2)^2} \simeq \delta^2$ is negligible

$$\Rightarrow \quad \boxed{E \simeq E_0\left(1 - \frac{E_0}{2mc^2}\right)}$$

(c) From part (b),

$$E_0 - E = \frac{E_0^2}{2mc^2}$$

Using $E = hf$, and writing $E_0^2 = hf_0 E_0$,

$$h(f_0 - f) = \frac{hf_0 E_0}{2mc^2}$$

$$\frac{\Delta f}{f_0} = \frac{E_0}{2mc^2} = \frac{4\times10^5\,eV}{2(2\times10^{11}\,eV)} = 10^{-6}$$

(d) For $v << c$, Doppler shift says

$$\frac{v}{c} \simeq \frac{\Delta\lambda}{\lambda} \qquad \text{where} \quad c = \lambda f$$

Since $f_0 > f$, the observer must move towards the photon

$$\Delta\lambda = c\,\Delta\left(\frac{1}{f}\right) \simeq -\frac{\Delta f}{f^2}\,c$$

Alternate:

$$c = \lambda f$$

$$dc = 0 = \lambda\,df + f\,d\lambda$$

$$d\lambda = -\lambda\frac{df}{f}$$

$$\frac{d\lambda}{\lambda} = -\frac{df}{f}$$

minus sign means that a decrease in λ corresponds to increase in frequency!

$\therefore$ (in magnitude)

$$\frac{v}{c} \simeq \frac{\Delta f}{f^2}\,\frac{c}{\lambda}$$

$$= \frac{\Delta f}{f}$$

$$v = \frac{\Delta f}{f_0}\,c$$

$$= 10^{-6}\,c \qquad \left(\text{using } \frac{\Delta f}{f_0} = 10^{-6} \text{ from part (c)}\right)$$

$$= 300 \text{ m/s}$$

(a) Use Gauss' law:

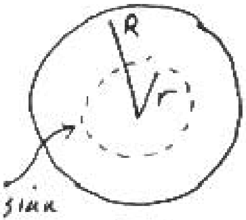
flux of $\vec{E}$ = $\frac{1}{\varepsilon_0}$ (charge enclosed)

$$\downarrow$$

$$E\, 2\pi r h = \frac{1}{\varepsilon_0}(n_0 e)(\pi r^2 h)$$

$$E = \left(\frac{n_0 e}{2\varepsilon_0}\right) r$$

$$\vec{E} = \frac{n_0 e}{2\varepsilon_0} r\,(-\hat{r}) \quad \text{radially } \underline{\underline{\text{in}}}$$



gaussian surface: cylinder, $r \le R$ height h

no flux through end caps ($\vec{E}\cdot\hat{n} = 0$ there)

(b)
$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

$$= -e\left[\frac{-n_0 e r\hat{r}}{2\varepsilon_0} + r\omega B_0 \hat{r}\right]$$

$\vec{B} = B_0 \hat{k}$

$\vec{v} = r\omega\hat{\theta}$

$\hat{\theta}\times\hat{k} = \hat{r}$

$$\vec{F} = \left(\frac{n_0 e^2 r}{2\varepsilon_0} - r\omega B_0 e\right)\hat{r}$$

(c)
$$\vec{F} = m\vec{a}$$

$$\left(\frac{n_0 e^2 r}{2\varepsilon_0} - r\omega B_0 e\right)\hat{r} = m\frac{(r\omega)^2}{r}(-\hat{r})$$

$$\Rightarrow \quad \frac{n_0 e^2}{2\varepsilon_0} - \omega B_0 e + m\omega^2 = 0$$

$$\omega^2 - \left(\frac{Be}{m}\right)\omega + \frac{n_0 e^2}{2\varepsilon_0 m} = 0$$

$$\omega^2 - \omega_c\omega + \tfrac{1}{2}\omega_p^2 = 0$$

$\omega_c \equiv \frac{B_0 e}{m}$

$\omega_p^2 \equiv \frac{n_0 e^2}{\varepsilon_0 m}$

$$\omega = \frac{\omega_c}{2} \pm \frac{1}{2}\sqrt{\omega_c^2 - 2\omega_p^2}$$

$$\omega = \frac{\omega_c}{2}\left(1 \pm \sqrt{1 - 2\left(\frac{\omega_p}{\omega_c}\right)^2}\right)$$

(d) Since ω is real, this requires

$$1 - 2\left(\frac{\omega_p}{\omega_c}\right)^2 \geq 0$$

$$\omega_p^2 \leq \frac{1}{2}\omega_c^2$$

$$\frac{n_0 e^2}{\varepsilon_0 m} \leq \frac{1}{2}\frac{e^2 B_0^2}{m^2}$$

$$n_0 \leq \frac{B_0^2 \varepsilon_0}{2m} \qquad \text{so} \quad (n_0)_{max} = \frac{B_0^2 \varepsilon_0}{2m}$$

(e) $$(n_0)_{max} = \frac{B_0^2 \varepsilon_0}{2m} \qquad \text{use} \quad \mu_0 \varepsilon_0 = \frac{1}{c^2}$$

$$= \frac{B_0^2}{2m(\mu_0 c^2)}$$

$$= \frac{B_0^2/2\mu_0}{mc^2} = \frac{\text{magnetic field energy density}}{\text{electron mass energy}}$$

Apply Ampere's law,

(e) $$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_S$$

where C = boundary of S

$$B'h = \mu_0 \int_S j\, da$$

where j = current density

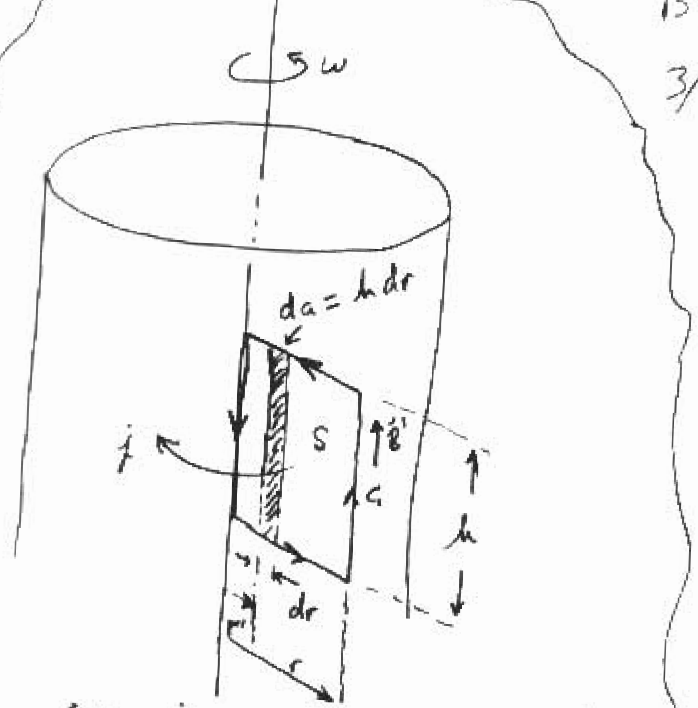
$$= (n_0 e)\, v$$
$$= n_0 e\, r'\omega$$

or $\Sigma r \Delta r$

$$B'h = \mu_0 (n_0 e \omega) \int_0^r r'(dr'\, h)$$

$$B'h = \mu_0 n_0 e \omega h \int_0^r r' dr'$$

$$B' = \mu_0 n_0 e \omega \frac{r^2}{2}$$



$\vec{B}'$ = induced field $= B'\hat{k}$

given $\vec{B}' = 0$ on axis

note $\vec{B}'\cdot d\vec{l} = 0$ horizontally

(f) Force due to induced magnetic field is $\vec{F}'$ where

$$\vec{F}' = -e\,\vec{v}\times\vec{B}' = -e(r\omega\hat{\theta})\times\left(\mu_0 n_0 e\omega \frac{r^2}{2}\hat{k}\right)$$

$$\hat{\theta}\times\hat{k} = \hat{r}$$

$$= \frac{\mu_0 n_0 e^2}{2}\omega^2 r^3 (-\hat{r})$$

$$v = r\omega$$

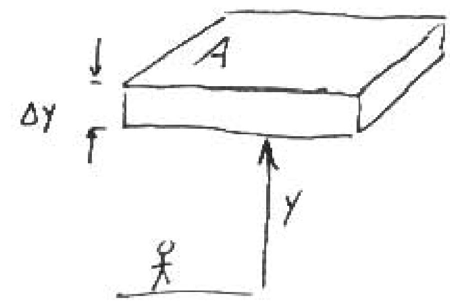
$$|\vec{F}'| = \frac{1}{2}\mu_0 n_0 e^2 v^2 r$$

$$\therefore \quad \frac{|\vec{F}'|}{|\vec{F}_{el.}|} = \frac{\frac{1}{2}\mu_0 n_0 e^2 v^2 r}{\frac{1}{2}\frac{n_0 e^2 r}{\varepsilon_0}} = \mu_0\varepsilon_0 v^2 = \frac{v^2}{c^2}$$

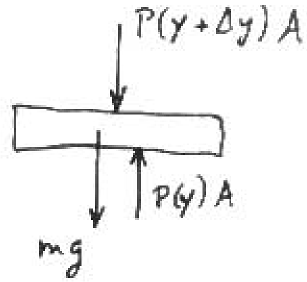
$|e\vec{E}|$ where $|\vec{E}|$ is from part (a)

(a) Apply $F = ma$ to a slab of atmosphere of thickness Δy, cross-sectional area A, density ρ. Take y positive upwards.

geometry:



Forces:



$$F = ma \quad (\text{which} = 0)$$

$$P(y)\,A - P(y+\Delta y)\,A - (\rho A \Delta y)\,g = 0$$

$$\underbrace{P(y+\Delta y) - P(y)}_{\Delta P} = -\rho g \,\Delta y$$

$$\Delta P = -\rho g \,\Delta y$$

note $\rho = \rho(y)$
$g = g(y)$

(b) Since $V_0 = \frac{-\Delta y}{\Delta t}$ (probe going down, y points up)

$$\Delta P = -\rho g \Delta y \quad \text{gives}$$

$$\frac{\Delta P}{\Delta t} = \rho g V_0 \qquad \ldots (1)$$

Need ρ: From ideal gas eq. of state $PV = nRT$,

$$P = \frac{n}{V} RT \qquad \text{where} \quad \frac{n}{V} = \frac{\#\text{ moles}}{\text{vol.}}$$

$$= \frac{\#\text{ moles}}{\text{mass}} \; \frac{\text{mass}}{\text{vol.}}$$

$$\frac{n}{V} = \frac{1}{M} \rho$$

$$\Rightarrow P = \frac{\rho}{M} RT$$

$$\rho = \frac{PM}{RT} \qquad \ldots (2)$$

Using ρ from (2) in (1),

$$\frac{\Delta P}{\Delta t} = \frac{PM}{RT} g V_0$$

$$V_0 = \left(\frac{\Delta P}{\Delta t}\right) \frac{RT}{PMg} \qquad \ldots (3)$$

From graph, probe lands on surface at $t = 4000$ s

$P = 60$ units

$g = 9.9$ N/kg

$T = 400$ K

From graph: at surface

$$\frac{\Delta P}{\Delta t} = \frac{(60-55)\text{ units}}{(4000-3900)\text{ s}} = .05 \text{ units/s}$$

[line from (3900, 55) to (4000, 60) is straight]

From Eq. (3),

$$v_0 = \frac{(.05 \text{ units/s})(8.3 \text{ J/K-mol})(400 \text{ K})}{(60 \text{ units})(44 \times 10^{-3} \text{ kg/mol})(9.9 \text{ N/kg})} \cong 6.4 \text{ m/s}$$

(c) Probe was 15 km above surface

$$\Delta t = \frac{15 \text{ km}}{6.4 \text{ m/s}} \cong 2344 \text{ s}$$ before touchdown; ie., at

$$t = (4000 - 2344)\text{ s} = 1656 \text{ s}$$

at $t \simeq 1650$ s, $P = 11$ units

slope at $t = 1650$ s is about the same as slope of line from (1500 s, 9 units) to (2000 s, 15 units)

$$\text{slope} \cong \frac{(15-9)\text{ units}}{(2000-1500)\text{ s}} = .012 \text{ units/s}$$

$\simeq .01$ to two sig. figures (like slope found above)

From Eq. (3), $T = \dfrac{PMgv_0}{R\left(\frac{\Delta P}{\Delta t}\right)}$

$$\Rightarrow \quad \frac{T_2}{T_1} = \frac{P_2}{P_1}\,\frac{(\Delta P/\Delta t)_1}{(\Delta P/\Delta t)_2}$$

Point 1: surface
Point 2: 15 km above surface

$$T_2 = \frac{11}{60}\left(\frac{.05}{.01}\right) 400\text{ K} \cong 367\text{ K}$$

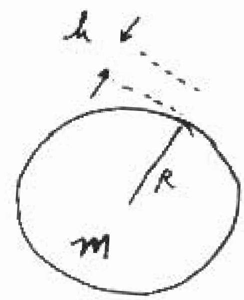
(d)

$$g(h) = \frac{GM}{(R+h)^2}$$

$$= \frac{GM}{R^2}\left(1+\frac{h}{R}\right)^{-2}$$

$$\cong g_0\left(1-\frac{2h}{R}\right)$$

$$\equiv g_0 - \Delta g$$



$g_0 = \dfrac{GM}{R^2}$

where $\Delta g = \dfrac{2h}{R}\, g_0$

$$\frac{\Delta g}{g_0} = \frac{2h}{R} = \frac{2\,(15\times10^3\text{ m})}{5\times10^6\text{ m}} = 6\times10^{-3}$$

In part (c), the estimate of P and $\Delta P/\Delta t$ at 15 km above the surface was much cruder than 6 parts out of a thousand.