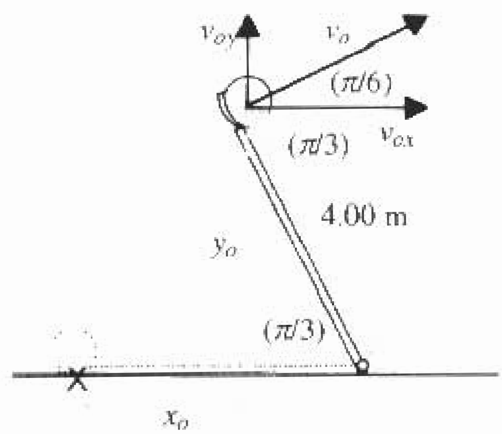
Solutions

(Suggested partial-credit points in square brackets at the right margin)

1. a. The tangential velocity of the rock as it leaves the catapult is the initial velocity of the projectile $v_o = v_T = 10.0\ \text{m/s}$ with $\theta_o = (\pi/6)$ [1+1]

First find the x- and y-components of the rock's position and velocity as it leaves the catapult arm and becomes a projectile. See the accompanying figure. Taking the point marked X as the origin



$$x_o = R(1-\cos(\pi/3)) = (4.00\ \text{m})(1-0.5) = 2.00\ \text{m} \quad [1]$$

$$y_o = R\sin(\pi/3) = (4.00\ \text{m})(0.866) = 3.46\ \text{m} \quad [1]$$

$$v_{ox} = v_o\cos(\pi/6) = (10.0\ \text{m/s})(0.866) = 8.66\ \text{m/s} \quad [1]$$

$$v_{oy} = v_o\sin(\pi/6) = (10.0\ \text{m/s})(0.500) = 5.00\ \text{m/s} \quad [1]$$

The equations for the x and y position of the rock are:

$$x = x_o + v_{ox}t \quad [2]$$

$$y = y_o + v_{oy}t + \left(\tfrac{1}{2}\right)a_y t^2 \quad [2]$$

with $a_y = -9.8\ \text{m/s}^2$ and $y = 0$ for the vertical position of the rocks landing point. [1+1]

Substituting $y = 0$ into the y-equation,

$$0 = \left(\tfrac{1}{2}\right)a_y t^2 + v_{oy}t + y_o \quad [1]$$

solving the quadratic equation for t and retaining only the positive t solution

$$t = \frac{-v_{oy} \pm \sqrt{v_{oy}^2 - 4\left(\tfrac{1}{2}\right)a_y y_0}}{2\left(\tfrac{1}{2}\right)a_y} = \frac{-5.00\ \text{m/s} \pm \sqrt{(5.00\ \text{m/s})^2 - 2(-9.8\ \text{m/s}^2)(3.46\ \text{m})}}{-9.8\ \text{m/s}^2} = 1.49\ \text{s} \quad [2]$$

$$x = x_o + v_{ox}t = 2.00\ \text{m} + (8.66\ \text{m/s})(1.49\ \text{s}) = 15\ \text{m} \quad [2]$$

b. The angular velocity of the rock at the point of release is:

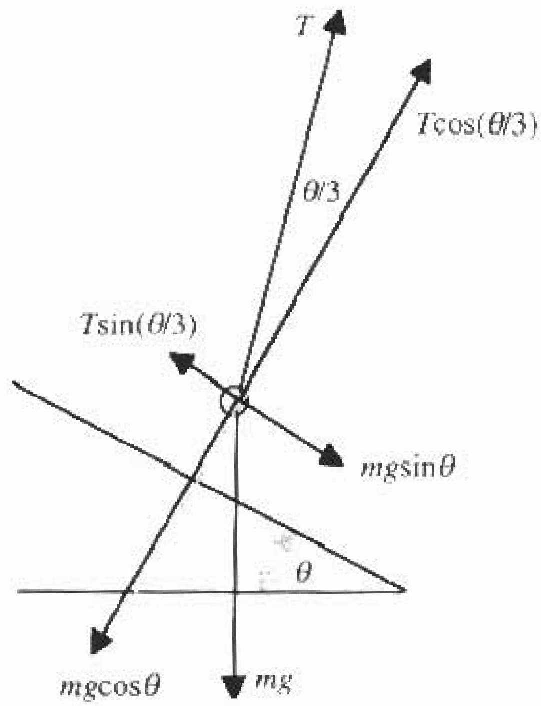
$$\omega = \frac{v_T}{R} = \frac{10.0\ \text{m/s}}{4.00\ \text{m}} = 2.50\ \text{rad/s} \quad [2]$$

Using $$\omega^2 = \omega_o^2 + 2\alpha\theta \quad [2]$$

with: $\omega_o = 0$ $\quad\theta = \left(\pi/3\right)$ rad [1+1]

$$\alpha = \frac{\omega^2 - \omega_o^2}{2\theta} = \frac{(2.5\ \text{rad/s})^2 - 0}{2\left(\pi/3\right)} = 2.98\ \text{rad/s}^2 \quad [2]$$

2. a. Resolving the string tension T and the ball weight mg into components parallel and perpendicular to the plane as shown in the diagram at the right,



{1 point for each correct force/component} [6]

we have for the perpendicular component: $\sum F_\perp = 0$ [1]

$$T\cos(\theta/3) = mg\cos\theta \qquad (2.1)$$ [1]

And for the parallel component: $\sum F_\parallel = ma$ [1]

$$mg\sin\theta - T\sin(\theta/3) = ma \qquad (2.2)$$ [1]

Solving (2.1) for T $\quad T = mg\dfrac{\cos\theta}{\cos(\theta/3)}$ [1]

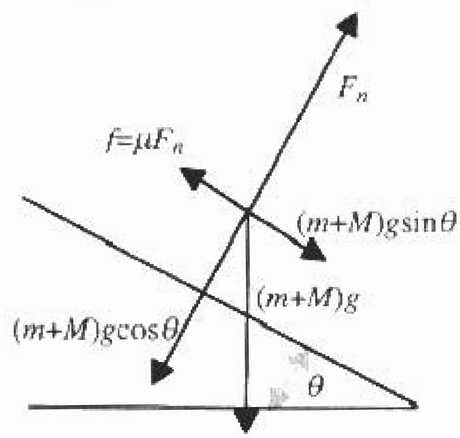
and substituting into (2.2):

$$mg\sin\theta - mg\frac{\cos\theta\sin(\theta/3)}{\cos(\theta/3)} = ma$$ [1]

$$a = g\left(\sin\theta - \cos\theta\frac{\sin(\theta/3)}{\cos(\theta/3)}\right) \qquad (2.3)$$ [1]

(While it is not required to simplify the above to obtain full credit, it equals $a = 2g\sin(\theta/3)$.)

b. The diagram to the right shows the forces acting on the cart-accelerometer system.



{1 point for each correct force/component} [5]

For the perpendicular component: $\sum F_\perp = 0$ [1]

$$F_n - (m+M)g\cos\theta = 0 \qquad (2.4)$$ [1]

And for the parallel component:

$$\sum F_\parallel = (m+M)a$$ [1]

$$(m+M)g\sin\theta - \mu F_n = (m+M)a \qquad (2.5)$$ [1]

Combining equations (2.3), (2.4), and (2.5)

$$(m+M)g\sin\theta - \mu(m+M)g\cos\theta = (m+M)g\left(\sin\theta - \cos\theta\frac{\sin(\theta/3)}{\cos(\theta/3)}\right)$$ [1]

Dividing through by $(m+M)g$

$$\sin\theta - \mu\cos\theta = \sin\theta - \cos\theta\frac{\sin(\theta/3)}{\cos(\theta/3)}$$ [1]

Solving for the coefficient of friction μ: $\quad \mu = \dfrac{\sin(\theta/3)}{\cos(\theta/3)} = \tan(\theta/3)$

(Or any equivalent answer.) [1]

3. a. The rotational inertia of a uniform thin rod about its end is found using the parallel axis theorem: $I = I_{cm} + M(L/2)^2 = \left(\frac{1}{12}\right)ML^2 + \left(\frac{1}{4}\right)ML^2 = \left(\frac{1}{3}\right)ML^2$ [2]

Using energy conservation: $K_o + U_o = K_F + U_F$ [2]

Let the lowest position of the rod's center of mass ($L/2$ below the axis) be the zero of gravitational potential energy. Then $U_o = Mg(L/2)$ and $U_F = 0$ [1+1]

For kinetic energy: $K_o = 0$ $K_F = (1/2)I\,\omega^2$ [1+1]

Substitution into the energy conservation equation yields

$$Mg(L/2) = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)ML^2\omega^2. \quad [1]$$

Dividing by M, L, and 1/2

$$g = \left(\frac{1}{3}\right)L\omega^2$$

or

$$\omega = \sqrt{\frac{3g}{L}} \quad [1]$$

b. The collision is inelastic. Using angular momentum conservation

$$I_o\omega_o = I_F\omega_F \quad [2]$$

Only the rod is moving initially, so $I_o = \left(\frac{1}{3}\right)ML^2$ and $\omega_o = \sqrt{\frac{3g}{L}}$ [1+1]

The final rotational inertia is the sum of the rod's and point mass's

$$I_F = \left(\frac{1}{3}\right)ML^2 + ML^2 = \left(\frac{4}{3}\right)ML^2 \quad [2]$$

Therefore:

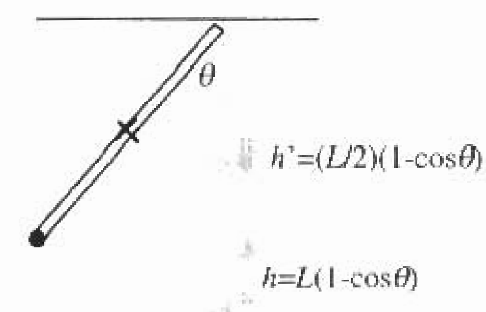
$$\omega_F = \omega_o\frac{I_o}{I_F} = \sqrt{\frac{3g}{L}}\left(\frac{\left(\frac{1}{3}\right)ML^2}{\left(\frac{4}{3}\right)ML^2}\right) = \left(\frac{1}{4}\right)\sqrt{\frac{3g}{L}} \quad [2]$$

c. Once the collision is over, mechanical energy is again conserved. Equating total mechanical energy immediately after the collision and at the highest point

$$K_o + U_o = K_F + U_F$$

At the highest point $K_F = 0$ [1]

$$K_o = \left(\frac{1}{2}\right)I\omega^2 = \left(\frac{1}{2}\right)\left(\frac{4}{3}\right)ML^2\left(\left(\frac{1}{4}\right)\sqrt{\frac{3g}{L}}\right)^2 = \left(\frac{1}{8}\right)MgL \quad [1]$$



Choosing the lowest point as the zero of gravitational potential energy, $U_o = 0$.

In swinging to an angle θ, the center of mass of the rod (marked by an x in the diagram) has moved up a distance $h' = (L/2)(1 - \cos\theta)$, [1]

While the point mass at the end has moved up $h = L(1 - \cos\theta)$. [1]

Therefore the final potential energy is

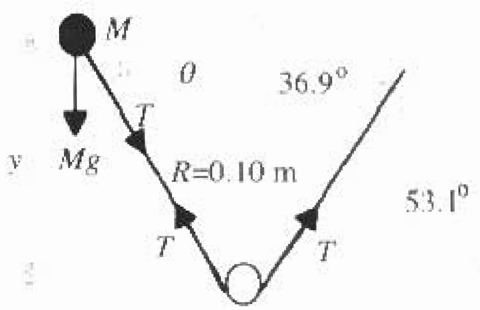
$$U_F = Mgh' + Mgh = Mg(L/2)(1-\cos\theta) + MgL(1-\cos\theta) = (3/2)MgL(1-\cos\theta). \quad [1]$$

Substituting into the energy conservation equation

$$\left(\tfrac{1}{8}\right)MgL = \left(\tfrac{3}{2}\right)MgL(1-\cos\theta) \quad [1]$$

$$\left(\tfrac{1}{12}\right) = 1-\cos\theta$$

$$\theta = \cos^{-1}\left(\tfrac{11}{12}\right) = 0.41 \text{ rad} \quad [1]$$



4. a. Applying energy conservation between the initial point $h_o = 0.15$ m and the point y

$$K_o + U_o = K_y + U_y \quad [2]$$

Initially	$K_o = 0$	$U_o = Mgh_o$	[1+1]
At point y	$K_y = \left(\tfrac{1}{2}\right)Mv^2$	$U_y = Mgy$	[1+1]

$$Mgh_o = \left(\tfrac{1}{2}\right)Mv^2 + Mgy \qquad (4.1) \quad [1]$$

The sphere is moving in a circle of radius

$$R = 0.60 \text{ m} - \sqrt{(0.40\text{ m})^2 + (0.30\text{ m})^2} = 0.10 \text{ m}. \quad [1]$$

The total centripetal force has two contributions, the tension in the string T and the radial component of the sphere's weight, $Mg\cos\theta$. Setting the total centripetal force equal to the mass times the centripetal acceleration

$$T + Mg\cos\theta = M\frac{v^2}{R} \qquad (4.2) \quad [4]$$

Solving (4.1) for Mv^2 and substituting into (4.2)

$$T + Mg\cos\theta = \frac{2Mg(h_o - y)}{R}. \qquad (4.3) \quad [1]$$

From the geometry $\cos\theta = \frac{y}{R}$. [1]

Combining with (4.3) and solving for T

$$T = \frac{2Mg(h_o - y)}{R} - Mg\frac{y}{R} = \frac{Mg(2h_o - 3y)}{R} = \frac{Mg(0.30m - 3y)}{0.10m}$$

$$T = 3Mg(1 - 10y)$$ where y is in meters. [1]

b. By Newton's third law, F, the force the rod exerts on the string, is equal and opposite to the force the string exerts on the rod. If there is no friction between the rod and the string, then the tension in both string segments is the same T.

$$F_x = -\sum T_{sx} = -\left(T\sin 36.9^o - T\sin\theta\right)$$ [2]

and $$F_y = -\sum T_{sy} = -\left(T\cos 36.9^o + T\cos\theta\right).$$ [2]

Since $\cos\theta = \frac{y}{R}$ $\sin\theta = \sqrt{1 - \left(\frac{y}{R}\right)^2}$ [1+1]

$$F_x = -T\left(0.60 - \sqrt{1 - \left(\frac{y}{R}\right)^2}\right)$$ [1]

$$F_x = -3Mg(1 - 10y)\left(0.60 - \sqrt{1 - (10y)^2}\right)$$ y in meters. [1]

$$F_y = -T\left(0.80 + \frac{y}{R}\right)$$ [1]

$$F_y = -3Mg(1 - 10y)(0.80 + 10y)$$ y in meters. [1]