AAPT | UNITED STATES PHYSICS TEAM
AIP | 2001

Solutions to Problems

Any correct solution should be awarded equivalent points. Suggested partial-credit points are presented in square brackets at the right margin. You may further break down the listed points into one point increments. If it is clear they have done an intermediate step, they should get credit for it even if they have not presented it. For example, in 1.a., if a student wrote down $mv_o r_o = mvr$, they should get 6 points credit. Students should not be penalized in a subsequent part for using the wrong answer to a previous part. (No double jeopardy.)

Points

1. a. The string force on the mass acts in the radial direction. It exerts no torque, so angular momentum is conserved. $L_0 = L$ [2]

For a particle of mass m moving with speed v in a circle of radius r, the angular momentum is

$$L = mvr.$$ [2]

Equating the angular momentum at time $t = 0$ to that at time t

$$mv_o r_o = mvr.$$ [2]

Solving for v $$v = \frac{v_o r_o}{r}.$$ Eq. 1-1 [1]

b. The string and attached mass are pulled with constant speed V radially inward

$$-V = \frac{\Delta r}{\Delta t} = \frac{r - r_o}{t - 0},$$ [2]

where the radially outward direction has been chosen to be positive.

$$r = r_o - Vt.$$ Eq. 1-2 [2]

Substituting this into Eq. 1-1 above $$v = \frac{v_o r_o}{(r_o - Vt)}$$ Eq. 1-3 [2]

c. Since the string is massless, the net force acting on it is zero. [1]

Denoting the force needed to pull the string by F and the tension in the string by T,

$$F = T,$$ [1]

The string tension T supplies the centripetal force acting on the mass

$$F = m\frac{v^2}{r}.$$ [2]

Substituting v and r from Eq. 1-3 and Eq. 1-2,

$$F = m\frac{v_o^2 r_o^2}{(r_o - Vt)^3}$$ [2]

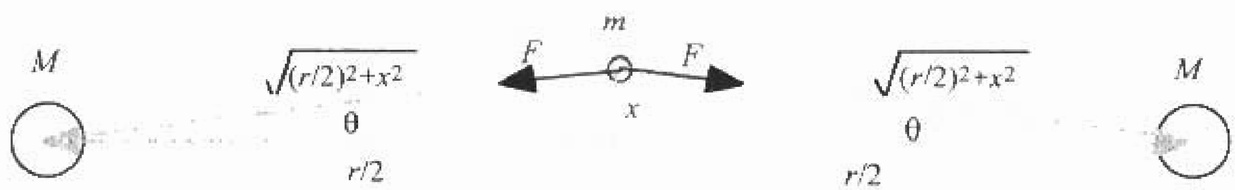
d. Using the work-energy theorem $W_{\text{net}} = \Delta K$ [2]

$$W_{net} = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_o^{\,2}$$ [2]

Combining with Eq. 1-3 $$W_{net} = \tfrac{1}{2}m\left(\frac{v_o r_o}{r_o - Vt}\right)^2 - \tfrac{1}{2}mv_o^{\,2} = \tfrac{1}{2}mv_o^{\,2}\left(\frac{r_o^{\,2}}{(r_o - Vt)^2} - 1\right)$$ [2]

2. *(If a student's only error is using r instead of r/2 as the asteroid to center of mass distance, deduct only one point for part a) and one point for part b).)*

a. Let x = the distance of the probe from the center of mass of the asteroid system. The diagram below is not to scale. The distance x has been greatly exaggerated. The probe is released from $A = x_{\max}$. Diagram [3]



The magnitude of the force F that each asteroid exerts on the probe is

$$F = \frac{GMm}{(r/2)^2 + x^2}.$$ Eq. 2-1 [2]

The total force on the probe is in the negative x-direction and equal to [1]

$$F_x = -2F\sin\theta$$ Eq. 2-2 [1]

with $$\sin\theta = \frac{x}{\sqrt{(r/2)^2 + x^2}}$$ Eq. 2-3 [1]

combining Eqs. 2-1, 2-2, 2-3 $$F_x = -2G\frac{Mm}{(r/2)^2 + x^2}\frac{x}{\sqrt{(r/2)^2 + x^2}} = -\frac{2GMmx}{\left((r/2)^2 + x^2\right)^{3/2}}$$ [1]

Since $x \le A << r$, $$\left((r/2)^2 + x^2\right)^{3/2} = \left((r/2)^2\right)^{3/2}\left(1 + (2x/r)^2\right)^{3/2} \approx (r/2)^3.$$ [1]

With this approximation, F_x becomes $$F_x \approx -\frac{16GMm}{r^3}x$$ [1]

Comparing this with the defining equation of simple harmonic motion $F = -kx$ [1]

(Students should be given equivalent credit for a valid method using calculus.)

The motion is approximately simple harmonic with

$$k = \frac{16GMm}{r^3}$$ Eq. 2-4 [1]

The period of simple harmonic motion is $T = 2\pi\sqrt{\frac{m}{k}}$ Eq. 2-5 [1]

Combining Eqs. 2-4 and 2-5 $T = 2\pi\sqrt{\frac{m}{(16GMm/r^3)}} = \frac{2\pi}{4}\sqrt{\frac{r^3}{GM}}$ [1]

The time t for the probe to reach the center of mass of the asteroid system is

$$t = \frac{T}{4}$$ [1]

$$t = \frac{2\pi}{16}\sqrt{\frac{r^3}{GM}} = \frac{\pi}{8}\sqrt{\frac{r^3}{GM}}$$ [1]

b. The center of mass of the asteroid system is the equilibrium position for the simple harmonic motion. The probe's velocity as it passes through the equilibrium position is

$$v_{\max} = \omega x_{\max} = \omega A$$ [2]

$$\omega = \frac{2\pi}{T}$$ [2]

$$\omega = \frac{2\pi}{\left(\frac{2\pi}{4}\right)\sqrt{\frac{r^3}{GM}}} = 4\sqrt{\frac{GM}{r^3}}$$ [2]

Then $$v_{\max} = 4A\sqrt{\frac{GM}{r^3}}$$ [2]

3. $m = 0.050$ kg $\quad mg = (0.050 \text{ kg})(10 \text{ m/s}^2) = 0.50$ N $\quad h = 1.47$ m $\quad f = 0.20$ N

a. Use the work energy theorem $\quad W_{nc} = \Delta K + \Delta U \quad$ Eq. 3-1 [2]

where W_{nc} is the work done by the non-conservative frictional force, ΔK is the change in kinetic energy, and ΔU is the change in potential energy. Let h' = greatest height after colliding with the floor. We will compare energies at the initial height h and the final height h'.

The frictional force is always opposite to the direction of motion, so the work done is the negative of the frictional force times the total distance traveled.

$$W_{nc} = -f(h + h')$$ [2]

$$\Delta K = 0$$ [1]

$$\Delta U = mg(h' - h)$$ [2]

Combining with Eq. 3-1 $$-f(h + h') = mg(h' - h)$$ [1]

and solving for h' $$(mg + f)h' = (mg - f)h$$

$$h' = h\left(\frac{mg - f}{mg + f}\right) = (1.47 \text{ m})\left(\frac{0.50 \text{ N} - 0.20 \text{ N}}{0.50 \text{ N} + 0.20 \text{ N}}\right) = 0.63 \text{ m}$$ [1]

b. As the bead moves downward, the frictional force is upward. Choosing the downward (acceleration) direction to be positive and the origin of the coordinate system to be at the point of release, we have

$$\sum F = ma.$$ [1]

$$mg - f = ma \qquad [1]$$

$$a = \frac{mg - f}{m} = \frac{0.50\text{ N} - 0.20\text{ N}}{0.050\text{ kg}} = 6.0\text{ m/s}^2 \qquad [1]$$

$$y = y_o + v_o t + \tfrac{1}{2}at^2 \qquad [1]$$

with $y = h = 1.47$ m, $y_o = 0$, and $v_o = 0$

$$t = \sqrt{\frac{2h}{a}} = \sqrt{\frac{2(1.47\text{ m})}{6.0\text{ m/s}^2}} = 0.70\text{ s} \qquad [2]$$

c. In order to find the initial velocity after the bounce, find the final velocity of the bead immediately before the bounce.

$$v = v_o + at = 0 + (6.0\text{ m/s}^2)(0.70\text{ s}) = 4.2\text{ m/s.} \quad \text{(downward)} \qquad [1]$$

Since the collision with the floor is elastic, the initial velocity of the bead after the bounce is

$$v_o = -4.2\text{ m/s} \quad \text{(upward)} \qquad [1]$$

As the bead moves upward $\sum F = ma$.

The frictional force is downward $mg + f = ma$ [1]

$$a = \frac{mg + f}{m} = \frac{0.50\text{ N} + 0.20\text{ N}}{0.050\text{ kg}} = 14.0\text{ m/s}^2 \qquad [1]$$

At the highest point, $v = 0$,

and
$$t = \frac{v - v_o}{a} = \frac{0 - (-4.2\text{ m/s})}{14.0\text{ m/s}^2} = 0.30\text{ s} \qquad [1]$$

d. Use the work energy theorem $W_{nc} = \Delta K + \Delta U$ [1]

where W_{nc} is the work done by the non-conservative frictional force, ΔK is the change in kinetic energy, and ΔU is the change in potential energy. We will compare energies at the initial height h and the final height 0. Let s = total distance traveled.

The work done by the frictional force is the negative of the frictional force times the total distance traveled.

$$W_{nc} = -fs \qquad [1]$$

$$\Delta K = 0$$

$$\Delta U = 0 - mgh \qquad [1]$$

Combining
$$-fs = -mgh \qquad [1]$$

and solving for s
$$s = \frac{mgh}{f} = \frac{(0.50\text{ N})(1.47\text{ m})}{0.20\text{ N}} = 3.7\text{ m} \qquad [1]$$

4. a. The two projectiles are simultaneously launched at $t = 0$ from $x_{2o} = 0$, $y_{2o} = 0$ and $x_{1o} = S$, $y_{1o} = 0$. Let u_o = the initial velocity of the second projectile. The components of the initial velocities are

$$v_{ox} = v_o \cos 45^o = \frac{v_o}{\sqrt{2}} \qquad v_{oy} = v_o \sin 45^o = \frac{v_o}{\sqrt{2}} \qquad [1]$$

$$u_{ox} = u_o \cos 30^o = u_o \frac{\sqrt{3}}{2} \qquad u_{oy} = u_o \sin 30^o = \frac{u_o}{2} \qquad [1]$$

Projectile 1 reaches its highest point when $v_y = 0$. [1]

$$v_y = v_{oy} - gt$$ [1]

or $$t = \frac{v_{oy}}{g} = \frac{v_o}{g\sqrt{2}}$$ [1]

At this time $x_1 = x_2$ and $y_1 = y_2$ [2]

with $$y = y_o + v_{oy}t - \tfrac{1}{2}gt^2$$ [1]

and $$x = x_o + v_{ox}t.$$ [1]

Equating y's at time t $$y_{1o} + v_{oy}t - \tfrac{1}{2}gt^2 = y_{2o} + u_{oy}t - \tfrac{1}{2}gt^2$$ [1]

$$v_{oy}t = u_{oy}t$$ [1]

Thus $$u_o = \sqrt{2}\, v_o.$$ [1]

Equating x's at time t $$x_{1o} + v_{ox}t = x_{2o} + u_{ox}t$$ [1]

Substituting in for t and v_o's $$S + \frac{v_o}{\sqrt{2}}\left(\frac{v_o}{g\sqrt{2}}\right) = 0 + u_o\frac{\sqrt{3}}{2}\left(\frac{v_o}{g\sqrt{2}}\right)$$

$$S + \frac{v_o}{\sqrt{2}}\left(\frac{v_o}{g\sqrt{2}}\right) = \sqrt{2}\, v_o\frac{\sqrt{3}}{2}\left(\frac{v_o}{g\sqrt{2}}\right)$$ [1]

$$S + \left(\frac{v_o^{\ 2}}{2g}\right) = \frac{\sqrt{3}}{2}\left(\frac{v_o^{\ 2}}{g}\right)$$

$$S = \left(\frac{v_o^{\ 2}}{2g}\right)\left(\sqrt{3} - 1\right)$$ [1]

b. The x- and y-components of the velocity are

$v_x = v_{ox}$ and $v_y = v_{oy} - gt$ [1]

Therefore $v_x = v_{ox} = \dfrac{v_o}{\sqrt{2}}$ and $v_y = v_{oy} - gt = \dfrac{v_o}{\sqrt{2}} - g\dfrac{v_o}{g\sqrt{2}} = 0$ [2]

and $$u_x = u_{ox} = u_o\frac{\sqrt{3}}{2} = \sqrt{2}v_o\frac{\sqrt{3}}{2} = v_o\sqrt{1.5}$$ [1]

$$u_y = u_{oy} - gt = \frac{u_o}{2} - g\frac{v_o}{g\sqrt{2}} = \frac{v_o\sqrt{2}}{2} - g\frac{v_o}{g\sqrt{2}} = 0$$ [1]

Both projectiles are at their highest points.

c. Since immediately before the collision the velocities are totally in the x-direction, the collision is one-dimensional. [1]
The collision is elastic so total momentum and total kinetic energy are unchanged [2]
Because the masses are equal, this requires the velocities to interchange.

$$v_{Fx} = v_o\sqrt{1.5}$$ [1]

$$u_{Fx} = \frac{v_o}{\sqrt{2}}$$ [1]