AAPT | UNITED STATES PHYSICS TEAM

AIP | 2001

2001 Semi-Final Exam
Part A – Solutions

A1. Each ball starts at rest and falls through a height H to its lowest point, which we will take as the zero of gravitational potential energy. Applying energy conservation, each ball reaches its lowest point with velocity

$$v_d = \sqrt{2gH}\,.$$

The large ball collides elastically with the floor and the direction of its velocity reverses. Moving upward, M collides elastically with m moving downward. Letting v be the speed of m and V be the speed of M after the collision and taking down to be negative and up positive, momentum conservation yields $\quad Mv_d - mv_d = mv + MV$.

Solving for V $\quad V = \dfrac{(Mv_d - mv_d - mv)}{M} = v_d - \dfrac{m}{M}(v_d + v)$

Energy conservation yields: $\quad \frac{1}{2}Mv_d^2 + \frac{1}{2}mv_d^2 = \frac{1}{2}MV^2 + \frac{1}{2}mv^2$.

Inserting V from the momentum equation,

$$\tfrac{1}{2}Mv_d^2 + \tfrac{1}{2}mv_d^2 = \tfrac{1}{2}M\left(v_d - \frac{m}{M}(v_d + v)\right)^2 + \tfrac{1}{2}mv^2$$

$$\tfrac{1}{2}Mv_d^2 + \tfrac{1}{2}mv_d^2 = \tfrac{1}{2}Mv_d^2 - \tfrac{1}{2}M2v_d\frac{m}{M}(v_d + v) + \tfrac{1}{2}M\frac{m^2}{M^2}\left(v_d^2 + 2v_dv + v^2\right) + \tfrac{1}{2}mv^2$$

$$\tfrac{1}{2}mv_d^2 = -\tfrac{1}{2}2v_dm(v_d + v) + \tfrac{1}{2}\frac{m^2}{M}\left(v_d^2 + 2v_dv + v^2\right) + \tfrac{1}{2}mv^2.$$

Multiplying by $2M/m$

$$Mv_d^2 = -2v_dM(v_d + v) + m\left(v_d^2 + 2v_dv + v^2\right) + Mv^2$$

Regrouping terms $\quad 0 = (m + M)v^2 - 2vv_d(M - m) + (m - 3M)v_d^2$.

Solving the quadratic equation for v yields

$$v = \left(\frac{M - m \pm 2M}{M + m}\right)v_d = \left(\frac{M - m \pm 2M}{M + m}\right)\sqrt{2gH}\,.$$

Since we want m moving upward (positive velocity), we select the plus sign.

$$v = \left(\frac{3M - m}{M + m}\right)\sqrt{2gH}\,.$$

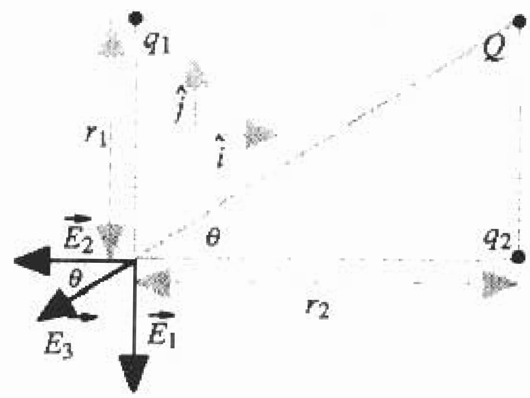
Let h equal the maximum height of the small ball above its lowest position. At m's highest point its kinetic energy is zero and

$$mgh = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\left(\frac{3M-m}{M+m}\right)\sqrt{2gH}\right)^2 = mgH\left(\frac{9M^2-6Mm+m^2}{M^2+2Mm+m^2}\right)$$

The ball's height above its initial position is $\Delta h = h - H$.

$$\Delta h = H\left(\frac{9M^2-6Mm+m^2}{M^2+2Mm+m^2}\right) - H = H\left(\frac{9M^2-6Mm+m^2-M^2-2Mm-m^2}{M^2+2Mm+m^2}\right)$$

$$\Delta h = H\left(\frac{8M^2-8Mm}{M^2+2Mm+m^2}\right) = H\frac{8M(M-m)}{(M+m)^2} = H\frac{8(1-m/M)}{(1+m/M)^2}$$



A2. (10) a. We will write the fields $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$ due to q_1, q_2, and Q, respectively, in terms of the usual unit vectors $\hat{i}$ and $\hat{j}$. See accompanying diagram.

$$\vec{E}_1 = -\hat{j}k\frac{q_1}{r_1^2}$$

where k = Coulomb's constant.

$$\vec{E}_2 = -\hat{i}k\frac{q_2}{r_2^2}$$

$$\vec{E}_3 = -\left(\hat{i}\cos\theta + \hat{j}\sin\theta\right)k\frac{Q}{\left(r_1^2+r_2^2\right)}.$$

Noting that $\cos\theta = \dfrac{r_2}{\sqrt{r_1^2+r_2^2}}$ and $\sin\theta = \dfrac{r_1}{\sqrt{r_1^2+r_2^2}}$, the total field becomes

$$\vec{E} = -k\left(\frac{q_2}{r_2^2} + \frac{Qr_2}{\left(r_1^2+r_2^2\right)^{3/2}}\right)\hat{i} - k\left(\frac{q_1}{r_1^2} + \frac{Qr_1}{\left(r_1^2+r_2^2\right)^{3/2}}\right)\hat{j}$$

(5) b. The field at X will vanish if each component vanishes.

$$\left(\frac{q_2}{r_2^2} + \frac{Qr_2}{\left(r_1^2+r_2^2\right)^{3/2}}\right) = 0 \quad \text{or} \quad q_2 = -\frac{Qr_2^3}{\left(r_1^2+r_2^2\right)^{3/2}}$$

$$\left(\frac{q_1}{r_1^2} + \frac{Qr_1}{\left(r_1^2+r_2^2\right)^{3/2}}\right) = 0 \quad \text{or} \quad q_1 = -\frac{Qr_1^3}{\left(r_1^2+r_2^2\right)^{3/2}}$$

(5) c. The electrostatic potential at X is

$$V = k\frac{q_1}{r_1} + k\frac{q_2}{r_2} + k\frac{Q}{\left(r_1^2+r_2^2\right)^{1/2}}.$$

For the charges found in Part b

$$V = -k\frac{Qr_1^2}{\left(r_1^2+r_2^2\right)^{3/2}} - k\frac{Qr_2^2}{\left(r_1^2+r_2^2\right)^{3/2}} + k\frac{Q}{\left(r_1^2+r_2^2\right)^{1/2}} = k\frac{Q\left(-r_1^2-r_2^2+r_1^2+r_2^2\right)}{\left(r_1^2+r_2^2\right)^{3/2}} = 0$$

(5) d. The total work is done by the electric force as a fourth charge q_4 is moved from infinity to the point marked X is $W = -\Delta U = -q_4 \Delta V = -q_4(V_X - V_\infty) = -q_4(0-0) = 0$

A3. (5) a. The resultant wave is the sum of the individual waves

$$\psi(z,t) = A\sin(kz_1 - \omega t + \phi) + A\sin(kz_2 - \omega t + \phi).$$

Using the trigonometric identity provided

$$\psi(z,t) = A2\cos\left[\tfrac{1}{2}\left((kz_1 - \omega t + \phi) - (kz_2 - \omega t + \phi)\right)\right]\sin\left[\tfrac{1}{2}\left((kz_1 - \omega t + \phi) + (kz_2 - \omega t + \phi)\right)\right]$$

$$\psi(z,t) = 2A\cos\left[\frac{k}{2}(z_1 - z_2)\right]\sin\left[\frac{k}{2}(z_1 + z_2) - \omega t + \phi\right].$$

(4) b. The amplitude of the resultant traveling wave is zero if $\cos\left[\frac{k}{2}(z_1 - z_2)\right] = 0.$

This occurs when the argument of the cosine is an odd half-integer multiple of π.

$$\frac{k}{2}\Delta z = \frac{2n+1}{2}\pi \qquad \text{for} \quad n = 0,1,2,\ ...$$

Since $k = \frac{2\pi}{\lambda}$ $\qquad \frac{1}{2}\frac{2\pi}{\lambda}\Delta z = \frac{2n+1}{2}\pi$.

or $\qquad \Delta z = \left(\frac{2n+1}{2}\right)\lambda \qquad \text{for} \quad n = 0,1,2,\ ...$

(4) c. The amplitude of the resultant traveling wave is maximum if $\cos\left[\frac{k}{2}(z_1 - z_2)\right] = \pm 1.$

This occurs when the argument of the cosine is an integer multiple of π.

$$\frac{k}{2}\Delta z = \frac{1}{2}\frac{2\pi}{\lambda}\Delta z = n\pi \qquad \text{for} \quad n = 0,1,2,\ ...$$

$$\Delta z = n\lambda \qquad \text{for} \quad n = 0,1,2,\ ...$$

(4) d. The frequency does not change as the wave enters the new material. Letting λ_m be the wavelength in the material

$$\frac{v_{so}}{\lambda} = \frac{v_{sm}}{\lambda_m}.$$

Solving for λ_m $\qquad \lambda_m = \frac{v_{sm}}{v_{so}}\lambda = \frac{1020\ \text{m/s}}{340\ \text{m/s}}\lambda = 3\lambda$

(8) e. The reflected sound intensity will be a minimum when the condition specified in part (b) is met. Δz is twice the thickness (up and down) of the coating material and λ is the wavelength in the material.

$$2t = \left(\frac{2n+1}{2}\right)\lambda_m \qquad \text{for} \quad n = 0,1,2,\ ...$$

$$2t = \left(\frac{2n+1}{2}\right)\frac{v_{sm}}{f}$$

or $$f=\left(\frac{2n+1}{2}\right)\frac{v_{sm}}{2t}=\left(\frac{2n+1}{2}\right)\frac{1020\text{ m/s}}{2(0.200\text{ m})}=(2n-1)1275\text{ Hz}$$

Inserting the integers into the above equation, we have

$$f = 1275\text{ Hz}, 3825\text{ Hz}, 6375\text{ Hz}, 8925\text{ Hz}$$

A4.(25 points) The sun's radius is $R_S = 6.96\text{x}10^8$ m and its surface temperature is $T = 5.80\text{x}10^3$ K. The sun-Earth distance is $R_{ES} = 1.50\text{x}10^{11}$ m. Each set of reactions release 2 neutrinos and thermal energy $Q = 26.2$ MeV = 26.2 MeV($1.6\text{x}10^{-13}$ J/MeV) = $4.192\text{x}10^{-12}$ J. The energy/neutrino ratio is $2.096\text{x}10^{-12}$ J/v For a perfect absorber emitter, $e = 1$.

The power radiated by the sun is $P = e\sigma AT^4 = e\sigma\left(4\pi R_S{}^2\right)T^4$.

The solar intensity reaching the Earth is

$$I=\frac{P}{A_{ES}}=\frac{e\sigma 4\pi R_S{}^2T^4}{4\pi R_{ES}{}^2}=\frac{(1)\left(5.67\times10^{-8}\text{ J}/(\text{s}\cdot\text{m}^2\cdot K^4)\right)\left(6.96\times10^8\text{ m}\right)^2(5800\text{ K})^4}{\left(1.50\times10^{11}\text{ m}\right)^2}=1.38\times10^3\ \frac{\text{J}}{\text{s}\cdot\text{m}^2}$$

To find the neutrino intensity I_v, divide the solar intensity I by the energy/neutrino ratio.

$$I_v=\frac{I}{Q/v}=\frac{1380\text{ J}/\left(\text{s}\cdot\text{m}^2\right)}{2.096\text{x}10^{-12}\text{ J}/v}=6.59\times10^{14}\text{ s}^{-1}m^{-2}$$

The neutrino intensity is also equal to their velocity c times their number density N/V. $I_v = cN/V$.

Solving for N, $$N=\frac{I_vV}{c}=\frac{\left(6.59\times10^{14}\text{ s}^{-1}m^{-2}\right)\left(8.00\times10^{-2}\text{ m}^3\right)}{3.0\times10^8\text{ m/s}}=1.76\times10^5$$

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Part B – Solutions

B1. a. (5) Using Gauss's Law the field is $E = k\dfrac{Q_{enc}}{r^2}$ where k is Coulomb's constant and Q_{enc} is the total charge enclosed by a Gaussian sphere of radius r. Since $b > r > a$, $Q_{enc} = +Q$. The field direction is radially outward and

$$\vec{E} = k\frac{Q}{r^2} \text{ radially outward}$$

b. (5) The total potential anywhere in the region $b \geq r \geq a$ is

$$V = k\frac{(-Q)}{b} + k\frac{Q}{r}$$

where the first term is due to the outer sphere and the second term to the inner sphere. finding the difference between V at $r = a$ and V at $r = b$,

$$V = \left(k\frac{(-Q)}{b} + k\frac{Q}{a}\right) - \left(k\frac{(-Q)}{b} + k\frac{Q}{b}\right) = kQ\left(\frac{1}{a} - \frac{1}{b}\right) \qquad \text{(B1-1)}$$

c. (5) The capacitance is defined $C = \dfrac{Q}{V}$.

Inserting equation (B1-1) $C = \dfrac{Q}{kQ\left(\frac{1}{a} - \frac{1}{b}\right)} = \dfrac{ab}{k(b-a)}$

d. (5) The work done in charging the capacitor to charge Q and voltage V is $W = \frac{1}{2}QV$. This is equal to the stored energy $U = \frac{1}{2}QV = \frac{1}{2}\dfrac{Q^2}{C} = \dfrac{k(b-a)}{2ab}Q^2$

e. (10) Since the capacitor has been disconnected from the battery without discharging the charge on the shells is still Q. Using the subscript K to denote quantities with the dielectric inserted, $Q_K = Q$.

In E, V, and C $k = \dfrac{1}{4\pi\varepsilon_o} \rightarrow \dfrac{1}{4\pi K\varepsilon_o} = \dfrac{k}{K}$

Therefore

$$E_K = E/K$$
$$V_K = V/K$$
$$C_K = KC$$

For the stored energy $U_K = \frac{1}{2} Q_K V_K = \frac{1}{2} Q \frac{V}{K} = \frac{U}{K}$

f. (5) The magnitude of the field at $b > r > a$ is $E_K = \frac{E}{K} = k\frac{(Q/K)}{r^2}$. Using Gauss's Law, the total charge enclosed by a Gaussian sphere of radius r is $Q_{enc} = Q/K$. This charge includes the charge on the conducting shell $+Q$ and the charge on the inner surface of the dielectric Q_D.

$$Q_{enc} = Q/K = +Q + Q_D$$

Solving for $Q_D = -\frac{Q(K-1)}{K}$

g. (10) Since the capacitor is left connected to the battery $V_K = V$.
Since V is unchanged so is E. $E_K = E$.
The capacitance is independent of whether or not the battery was connected. $C_K = KC$
Additional charge flows onto the capacitor $Q_K = C_K V_K = KCV = KQ$
The energy stored is $U_K = \frac{1}{2} Q_K V_K = \frac{1}{2} KQV = KU$
Applying Gauss's Law to find the total charge enclosed by a Gaussian sphere with radius r,

$$Q_{enc} = Q = Q_K + Q_D = KQ + Q_D$$

Solving for $Q_D = -Q(K-1)$

h. (5) The two halves can be considered capacitors in parallel. Each has half the capacitance it would have if it were whole. Therefore $C = C_1 + C_2 = \frac{1}{2}\frac{K_1 ab}{k(b-a)} + \frac{1}{2}\frac{K_2 ab}{k(b-a)} = \frac{1}{2}\frac{(K_1 + K_2)ab}{k(b-a)}$

B2. a. (5) The magnetic force is $\vec{F} = q\vec{v} \times \vec{B}$.
Inserting the expressions for the velocity and magnetic field and taking the cross product,

$$\vec{F} = q(v_r\hat{u}_r + r\omega\hat{u}_\theta) \times B_o\hat{u}_z = qB_o(v_r\hat{u}_r \times \hat{u}_z + r\omega\hat{u}_\theta \times \hat{u}_z)$$

$$\vec{F} = qB_o(r\omega\hat{u}_r - v_r\hat{u}_\theta)$$

b. (5) At any instant the displacement is in the direction of the velocity while the force is perpendicular to it. Therefore $W = 0$.
Or $dW = \vec{F} \cdot d\vec{l} = \vec{F} \cdot \vec{v}dt = q(\vec{v} \times \vec{B}) \cdot \vec{v}dt = 0$

c. (8) The magnetic force does no work. The other force has an associated potential energy. Using the work-energy theorem which reduces to conservation of mechanical energy,

$$U_o + K_o = U + K$$

Since the particle is initially at rest, its initial kinetic energy is zero. $K_o = 0$
Substituting the energy terms

$$-\frac{k}{r_o^2} = \frac{1}{2}mv^2 - \frac{k}{r^2},$$

Where $v^2 = (v_r\hat{u}_r + r\omega\hat{u}_\theta) \cdot (v_r\hat{u}_r + r\omega\hat{u}_\theta) = v_r^2 + r^2\omega^2$

Combining the last two equations

$$-\frac{k}{r_o^2} = -\frac{k}{r^2} + \frac{1}{2}m\left(v_r^2 + r^2\omega^2\right) \qquad \text{(B2-1)}$$

d. (7) The torque about the z-axis is $\vec{\tau} = \vec{r} \times \vec{F} = r\hat{u}_r \times \vec{F}$. The other force $\vec{F} = -\frac{2k}{r^3}\hat{u}_r$ contributes no torque, since $\hat{u}_r \times \hat{u}_r = 0$. Therefore the net torque is due to the magnetic force

$$\vec{\tau} = r\hat{u}_r \times qB_o\left(r\omega\hat{u}_r - v_r\hat{u}_\theta\right) = -qB_o r v_r \hat{u}_z$$

e. (5) The net torque is equal to the time rate of change of angular momentum. The radial component of the velocity v_r is equal to the time rate of change of r.

$$\frac{d\vec{L}}{dt} = \vec{\tau} = -qB_o r v_r \hat{u}_z = -qB_o r\frac{dr}{dt}\hat{u}_z = -qB_o \frac{1}{2}\frac{dr^2}{dt}\hat{u}_z$$

$$\frac{d}{dt}\left(\vec{L} + qB_o\frac{1}{2}r^2\hat{u}_z\right) = 0$$

Or

$$\vec{L} + qB_o\frac{1}{2}r^2\hat{u}_z = \vec{C},$$

where $\vec{C}$ is a constant vector.

$$\vec{L} = \vec{C} - qB_o\frac{1}{2}r^2\hat{u}_z.$$

Dropping the vector notation since all motion occurs in the xy-plane,

$$L = C - qB_o\frac{1}{2}r^2$$

f. (5) For a particle the magnitude of the angular momentum about the z-axis can be written

$$L = mr^2\omega.$$

Combining the last two equations

$$mr^2\omega = C - qB_o\frac{1}{2}r^2$$

Or

$$mr^2\omega + qB_o\frac{1}{2}r^2 = C.$$

The constant C can be evaluated in terms of the initial conditions

$$mr^2\omega + qB_o\frac{1}{2}r^2 = mr_o^2\omega_o + qB_o\frac{1}{2}r_o^2$$

since the particle is initially at rest $\omega_o = 0$ and

$$mr^2\omega + qB_o\frac{1}{2}r^2 = qB_o\frac{1}{2}r_o^2$$

Solving for ω, we have

$$mr^2\omega = qB_o\frac{1}{2}\left(r_o^2 - r^2\right)$$

$$\omega = \frac{qB_o}{2m}\left(\frac{r_o^2}{r^2} - 1\right) \qquad \text{(B2-2)}$$

g. (5) At $r_{\min}$, $v_r = 0$, Equation (B2-1) becomes

$$-\frac{k}{r_o^2} = -\frac{k}{r_{\min}^2} + \frac{1}{2}mr_{\min}^2\omega^2$$

Bring the other k term to the left and substituting equation (B2-2) at $r_{\min}$

$$\frac{k}{r_{\min}^2} - \frac{k}{r_o^2} = \frac{1}{2} m r_{\min}^2 \left(\frac{qB_o}{2m} \left(\frac{r_o^2}{r_{\min}^2} - 1 \right) \right)^2$$

or multiplying by $r_{\min}^2 r_o^2$, $k\left(r_o^2 - r_{\min}^2\right) = \frac{r_o^2 q^2 B_o^2}{8m} \left(r_o^2 - r_{\min}^2\right)^2$.

Canceling the common factor $k = \frac{r_o^2 q^2 B_o^2}{8m} \left(r_o^2 - r_{\min}^2\right)$

and solving for $r_{\min}^2 = r_o^2 - \frac{8km}{q^2 B_o^2 r_o^2}$ (B2-3)

h. (5) The particle will not reach the origin if $r_{\min}^2 > 0$. From the preceding expression, this occurs when

$$r_o^2 > \frac{8km}{q^2 B_o^2 r_o^2},$$

or solving for

$$B_o^2 > \frac{8km}{q^2 r_o^4}.$$

Thus

$$\left|B_o\right| > \frac{1}{r_o^2} \sqrt{\frac{8km}{q^2}}$$

i. (5) Evaluating equation (B2-2)

at $r = r_o$ $\omega_o = \frac{qB_o}{2m} \left(\frac{r_o^2}{r_o^2} - 1 \right) = 0$

at $r = r_{\min}$ $\omega = \frac{qB_o}{2m} \left(\frac{r_o^2}{r_{\min}^2} - 1 \right)$.

combining with equation (B2-3) $\omega = \frac{qB_o}{2m} \left(\frac{r_o^2}{r_o^2 - \frac{8km}{q^2 B_o^2 r_o^2}} - 1 \right) = \frac{qB_o}{2m} \left(\frac{1}{1 - \frac{8km}{q^2 B_o^2 r_o^4}} - 1 \right)$.

For large B_o, $\frac{8km}{q^2 B_o^2 r_o^4}$ is small. Using the binomial expansion

$$\omega = \frac{qB_o}{2m} \left(1 + \frac{8km}{q^2 B_o^2 r_o^4} - 1 \right) = \frac{4k}{qB_o r_o^4}$$