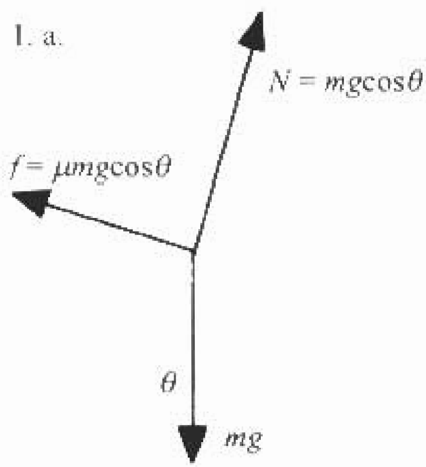
AAPT | UNITED STATES PHYSICS TEAM

AIP | 2003

Solutions to Problems

Unless explicitly told to use a specific method, any correct solution should be awarded equivalent points. Suggested partial-credit points are presented in square brackets at the right margin. You may further break down the listed points into one point increments. If it is clear they have done an intermediate step, they should get credit for it even if they have not presented it. For example, in 2.a., if a student wrote down $2v_o t - \frac{1}{2}gt^2 = H - v_o t - \frac{1}{2}gt^2$, they should get 8 points credit. Students should not be penalized in a subsequent part for using the wrong answer to a previous part. (No double jeopardy.)

1. a.



Points

Drawing and labeling the mg vector in the downward direction <u>or</u> correctly drawing and labeling the components parallel to the plane $(mg\sin\theta)$ and perpendicular to the plane $(mg\cos\theta)$. [3]

Drawing and labeling the normal force (any alternate symbol for N may be used).

$$N = mg\cos\theta.$$ [3]

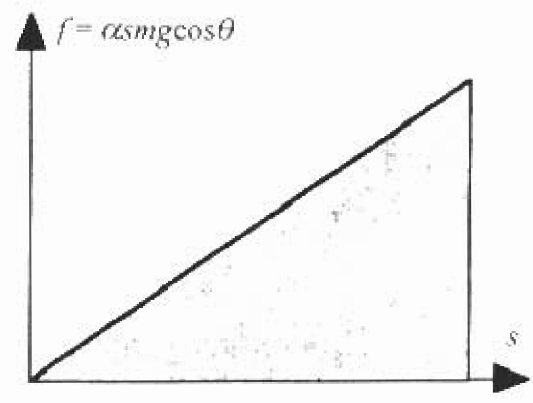
Drawing and labeling the friction force (any alternate symbol for f may be used).

$$f = \mu mg\cos\theta$$

or $$f = \alpha smg\cos\theta.$$ [3]

b. Sketch of force versus displacement graph showing a linear function of s. [3]

b.



c. Finding the area under the graph by any method and equating it to $|W|$. For example,

$$|W| = Area = \tfrac{1}{2}bh = \tfrac{1}{2}(s)(\alpha smg\cos\theta)$$

$$|W| = \tfrac{1}{2}(\alpha mg\cos\theta)s^2.$$ [3]

{The absolute value signs around W are not required.}

d. Using the work-energy theorem

$$PE_i + KE_i + W = PE_f + KE_f$$ [1]

"i" refers to the starting point and "f" refers to the stopping point, a distance s down the plane. At both points the kinetic energy is zero

$KE_i = 0$ $KE_f = 0.$ [2]

Setting the potential energy at the stopping point equal to zero, the potential energies are

$$PE_i = mgh \qquad PE_f = 0. \qquad [2]$$

where h is the vertical displacement of the starting point above the stopping point

$$h = s\sin\theta. \qquad [1]$$

The frictional force does negative work

$$W = -\tfrac{1}{2}(\alpha mg\cos\theta)s^2. \qquad [1]$$

Combining terms: $$mgs\sin\theta - \tfrac{1}{2}(\alpha mg\cos\theta)s^2 = 0 \qquad [1]$$

The $s = 0$ solution represents the starting point and can be eliminated.

Dividing by mgs, $$\sin\theta - \tfrac{1}{2}(\alpha\cos\theta)s = 0 \qquad [1]$$

Therefore $$s = \frac{2\sin\theta}{\alpha\cos\theta} = \frac{2}{\alpha}\tan\theta \qquad [1]$$

2. a. Selecting down to be negative and the ground to be the zero of position, the position of Galileo's cannonball is

$$y_G = H - v_o t - \tfrac{1}{2}gt^2. \qquad (2\text{-}1) \qquad [3]$$

The position of his friend's cannonball is

$$y_F = 0 + 2v_o t - \tfrac{1}{2}gt^2. \qquad (2\text{-}2) \qquad [3]$$

The cannonballs collide when they are at the same position at the same time. Setting the equations equal

$$2v_o t - \tfrac{1}{2}gt^2 = H - v_o t - \tfrac{1}{2}gt^2. \qquad [2]$$

Solving for t, $$3v_o t = H,$$

or $$t = \frac{H}{3v_o}. \qquad (2\text{-}3) \qquad [2]$$

b. The velocities at the time of collision are:

for Galileo's cannonball $$v_G = -v_o - gt \qquad [3]$$

and for the friend's cannonball $$v_F = 2v_o - gt. \qquad [3]$$

The velocities are related $$v_G = 7v_F. \qquad [2]$$

Combining $$-v_o - gt = 7(2v_o - gt)$$

or $$6gt = 15v_o. \qquad (2\text{-}4) \qquad [2]$$

There are many ways of combining equations (2-3) and (2-4) with either (2-1) or (2-2) to get y in terms of only H. One of the shortest is to multiply (2-4) by t and use (2-3) to simplify.

$$6gt^2 = 15v_o t = 15v_o\left(\frac{H}{3v_o}\right) = 5H \qquad [3]$$

Subbing into equation (2-2), the collision occurs at

$$y_F = 0 + 2v_o t - \tfrac{1}{2}gt^2 = 2\left(\frac{H}{3}\right) - \tfrac{1}{2}\left(\frac{5H}{6}\right) = \frac{H(8-5)}{12} = \frac{H}{4}. \qquad [2]$$

3. a. Using energy conservation $$PE_i + KE_i = PE_f + KE_f \qquad [1]$$

"i" refers to the starting point at the top of the plane and "f" refers to the bottom of the plane.

Taking the bottom of the planes as our zero of potential energy $PE_f = 0.$ [1]

At the top of the plane $PE_i = MgH.$ $KE_i = 0.$ [1]

At the bottom of the plane, the kinetic energy consists of translational kinetic energy plus rotational kinetic energy

$$\mathrm{KE}_f = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2. \quad [2]$$

Substituting into the energy conservation equation

$$MgH = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2. \quad (3\text{-}1)$$

Since it rolls without slipping $\omega = \dfrac{v}{R}$. [1]

Combining terms in Eq (3-1)

$$MgH = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}CMR^2\left(\frac{v}{R}\right)^2 = \tfrac{1}{2}Mv^2(1+C). \quad [1]$$

Solving for v $v = \sqrt{\dfrac{2gH}{1+C}}$. [1]

b. By the parallel axis theorem, the moment of inertia about a point on the rim is

$$I = I_{cm} + MR^2 = CMR^2 + MR^2 = (1+C)MR^2. \quad [1]$$

Either remembering or deriving the equation for the period of a physical pendulum

$$T = 2\pi\sqrt{\frac{I}{Mgd}} \quad [4]$$

where I is the moment of inertia about the pivot point and d is the distance between the pivot point and the center of mass $d = R$. [1]

Combining these equations

$$T = 2\pi\sqrt{\frac{(1+C)MR^2}{MgR}} = 2\pi\sqrt{\frac{(1+C)R}{g}}. \quad [1]$$

c. Let σ = the mass per unit area in the x-y plane. Consider a solid disk of radius R and density σ and four solid disks with radius $R/4$ and density $-\sigma$.

The mass of a solid disk is $M_{SD} = \sigma\pi R^2$. [1]

The moment of inertia of the solid disk is $I_{SD} = \tfrac{1}{2}M_{SD}R^2 = \tfrac{1}{2}(\sigma\pi R^2)R^2 = \tfrac{1}{2}\sigma\pi R^4$. [1]

The mass of a hole is $M_H = -\sigma\pi\left(\dfrac{R}{4}\right)^2$. [1]

The moment of inertia a hole about its center $I_{H,cm} = \tfrac{1}{2}M_H\left(\dfrac{R}{4}\right)^2$. [1]

Using the parallel axis theorem to find the moment of inertia of a hole about the center of the disk, a distance $d = R/2$ from the center of the hole,

$$I_H = I_{H,cm} + M_H d^2 = \tfrac{1}{2}M_H\left(\frac{R}{4}\right)^2 + M_H\left(\frac{R}{2}\right)^2 = \frac{1}{32}M_H R^2(1+8) = \frac{9}{32}M_H R^2. \quad [1]$$

$$I_H = \frac{9}{32}\left(-\sigma\pi\left(\frac{R}{4}\right)^2\right)R^2 = -\frac{9}{512}\sigma\pi R^4. \quad [1]$$

The total mass of the disk

$$M = M_{SD} + 4M_H = \sigma\pi R^2 + 4\left(-\sigma\pi\left(\frac{R}{4}\right)^2\right) = \sigma\pi R^2\left(1 - 4\left(\frac{1}{16}\right)\right) = \frac{3}{4}\sigma\pi R^2. \quad [2]$$

The total moment of inertia of the disk is

$$I = I_{SD} + 4I_H = \frac{1}{2}\sigma\pi R^4 + 4\left(-\frac{9}{512}\sigma\pi R^4\right) = \frac{1}{128}\sigma\pi R^4(64-9) = \frac{55}{128}\sigma\pi R^4.$$

Rewriting it in terms of M $$I = \frac{55}{128}\sigma\pi R^4\left(\frac{M}{\frac{3}{4}\sigma\pi R^2}\right) = \frac{55}{128}\frac{4}{3}MR^2 = \frac{55}{96}MR^2.$$ [1]

To find C $$C = \frac{I}{MR^2} = \frac{55}{96}$$ [1]

4. a. When the cylinder is first placed on the conveyor belt, it is slipping and will experience a force of kinetic friction f. This causes both an acceleration of the center of mass and an angular acceleration about the center of mass. The cylinder will continue to accelerate until it has reached a velocity at which it rolls without slipping. Then the net force is zero. [2]

When it rolls without slipping, $v = v_{cm} + v_{rel}$ (4-1) [2]

where v_{cm} is the final velocity of the center of mass and

$$v_{rel} = \omega R$$ [1]

is the velocity of a point on the rim relative to the center of mass.

{There are at least three methods that can be used to find the relation between v_{cm} and ωR. These are detailed below. Students may be awarded points from only one method.}

Method I – Using the acceleration to find the velocities.

{It is not necessary to set $f = \mu Mg$. In all of the following answers μg and f / M may be used interchangeably.} Finding the translational acceleration of the center of mass

$$\sum F = Ma_{cm}$$ [1]

$$f = \mu Mg = Ma_{cm}$$

or $$a_{cm} = \mu g.$$ [1]

Since it starts from rest, after time t, the velocity of the center of mass is

$$v_{cm} = \mu gt.$$ [1]

Finding the angular acceleration about the center of mass

$$\sum \tau = I\alpha.$$ [1]

The torque is $\tau = Rf = \mu RMg.$ [1]

The moment of inertia of a solid uniform cylinder is

$$I = \tfrac{1}{2}MR^2$$ [1]

$$\alpha = \frac{\tau}{I} = \frac{fR}{\frac{1}{2}MR^2} = \frac{2\mu Mg}{MR} = \frac{2\mu g}{R}.$$ [1]

After time t, the angular velocity is $\omega = \alpha t = \frac{2\mu gt}{R}.$ [1]

The velocity of a point on the rim relative to the center of mass is

$$\omega R = 2\mu gt = 2v_{cm}.$$ [1]

Method II – Using impulse to find the velocities.

The magnitude of the impulse is given by $\Delta p_{cm} = F_{net}t.$ [1]

where t is the time for which the force acts. The change in momentum is

$$\Delta p_{cm} = Mv_{cm} - 0.$$ [1]

$$Mv_{cm} = ft.$$ (4–2) [1]

The magnitude of the angular impulse is $\Delta L = \tau t.$ [1]

The change in angular momentum is $\Delta L = I\omega - 0.$ [1]

The magnitude of the torque is $\tau = fR.$ [1]

Combining $I\omega = fRt$. (4-3) [1]

For a cylinder the moment of inertia is $I = \frac{1}{2}MR^2$. (4-4) [1]

Substituting Eq. (4-2) and (4-4) into (4-3)

$$\tfrac{1}{2}MR^2\omega = Mv_{cm}R. \quad [1]$$

Solving for ωR $\quad \omega R = 2v_{cm}$.

<u>Method III.</u> Use conservation of angular momentum.

This is easiest to do in the lab frame. Consider an axis that is at rest in the lab frame and lies along the initial line of contact between the cylinder and the belt. The line of action of the friction force passes through this axis, so friction provides no torque. The normal force and the weight force provide equal and opposite torques about any axis. The net torque about this axis is zero.

$$\sum \tau = 0. \quad [2]$$

The total angular momentum equals a constant C $\quad \sum L = C$. [1]

The angular momentum has two components, the angular momentum due to the rotation of the cylinder and the angular momentum due to the translation of the center of mass.

$$\sum L = L_{rot} + L_{trans}.$$

These are in equal and opposite senses $\quad \sum L = I\omega - MRv_{cm} = C$. [2]

Since $\omega = 0$ and $v_{cm} = 0$ initially, $C = 0$. [1]

$$I\omega = MRv_{cm} \quad [1]$$

For a cylinder the moment of inertia is $I = \frac{1}{2}MR^2$. [1]

Combining $\frac{1}{2}MR^2\omega = Mv_{cm}R$. [1]

Solving for ωR $\quad \omega R = 2v_{cm}$.

<u>Then for all methods:</u>

Substituting into Eq. (4.1) $\quad v = v_{cm} + \omega R$.

$$v = v_{cm} + 2v_{cm}.$$

So $\quad v_{cm} = \frac{1}{3}v$. [1]

b. The velocity does not depend on M or R, therefore

$$v_{cm} = \tfrac{1}{3}v. \quad [5]$$

c. If a ring of infinitesimal thickness were used, the moment of inertia and angular velocity would change. Now

$$I = MR^2, \quad [2]$$

$$v_{rel} = \omega R = \alpha t R = \frac{\tau}{I} tR = \frac{\mu RMg}{MR^2} tR = \mu g t \quad [1]$$

$$v = v_{cm} + v_{rel} = \mu g t + \mu g t$$

$$v_{cm} = \tfrac{1}{2}v. \quad [2]$$