AAPT UNITED STATES PHYSICS TEAM
AIP 2006

Solutions to Problems

Each question is worth 25 points. Any correct solution should be awarded equivalent points. Suggested partial-credit points are presented in square brackets in the right margin after the equation number. You may further break down the listed points into one point increments. If it is clear they have done an intermediate step, they should get credit for it even if they have not presented it. Students should not be penalized in a subsequent part for using the wrong answer to a previous part. (No double jeopardy.)

Question 1

There are at least two ways to approach this problem. Other solution methods may exist as well.

Method 1

Here the focus is on angular rotation variables.

Let $kr = 2.0\ \mathrm{m/s^3}$. Then the angular acceleration is given by

$$\alpha_t = kt. \qquad (1\text{-}1) \quad [\mathbf{3\ pt}]$$

and the angular velocity is given by

$$\omega = \int_0^t \alpha_t\, dt = \frac{1}{2}kt^2, \qquad (1\text{-}2) \quad [\mathbf{3\ pt}]$$

and the angular position relative to P is given by

$$\theta = \int_0^t \omega\, dt = \frac{1}{6}kt^3. \qquad (1\text{-}3) \quad [\mathbf{3\ pt}]$$

The velocity is always tangential, the acceleration has tangential and radial components, so the tangent of the angle ϕ between $\vec{v}$ and $\vec{a}$ is given by

$$\tan\phi = a_r/a_t. \qquad (1\text{-}4) \quad [\mathbf{2\ pt}]$$

Now

$$a_r = r\omega^2, \tag{1-5}$$ **[2 pt]**

and

$$a_t = r\alpha, \tag{1-6}$$ **[2 pt]**

so combining Eq. 1-1 through Eq. 1-6,

$$\begin{aligned}
\tan\phi &= \frac{a_r}{a_t}, && (1\text{-}7)\\
&= \frac{r\omega^2}{r\alpha}, && (1\text{-}8)\\
&= \frac{\left(\frac{1}{2}kt^2\right)^2}{kt}, && (1\text{-}9)\\
&= \frac{1}{4}kt^3, && (1\text{-}10)\\
&= \frac{3}{2}\theta. && (1\text{-}11)
\end{aligned}$$ **[6 pt]**

Then

$$\begin{aligned}
\theta &= \frac{2}{3}\tan\phi, && (1\text{-}12)\\
&= \frac{2}{9}\sqrt{3}. && (1\text{-}13)
\end{aligned}$$ **[4 pt]**

Method 2

Here the focus is on linear variables.

Let $c = 2.0\ \mathrm{m/s^3}$. Then the tangential acceleration is given by

$$a_t = ct, \tag{1-14}$$ **[3 pt]**

and the tangential velocity is given by

$$v = \int_0^t a_t\, dt = \frac{1}{2}ct^2, \tag{1-15}$$ **[3 pt]**

and the arc length s traveled in time t is given by

$$s = \int_0^t v\, dt = \frac{1}{6}ct^3. \tag{1-16}$$ **[3 pt]**

The radial acceleration is found from Eq. 1-15,

$$a_r = \frac{v^2}{r} = \frac{c^2t^4}{4r}. \tag{1-17}$$ **[2 pt]**

The angle θ between the velocity vector and the acceleration vector is found from the dot product:

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{a}} = va\cos\phi, \qquad (1\text{-}18) \quad \textbf{[2 pt]}$$

where v and a are the magnitudes of the velocity and acceleration. In unit vector notation,

$$\vec{\mathbf{a}} = a_t\boldsymbol{\theta} + a_r\hat{\mathbf{r}}, \qquad (1\text{-}19)$$

$$\vec{\mathbf{v}} = v\boldsymbol{\theta}, \qquad (1\text{-}20)$$

so

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{a}} = va_t, \qquad (1\text{-}21)$$

$$= vct. \qquad (1\text{-}22) \quad \textbf{[2 pt]}$$

The magnitude of the acceleration vector found from Eq. 1-14 and Eq. 1-17,

$$a = \sqrt{{a_r}^2 + {a_t}^2}, \qquad (1\text{-}23)$$

$$= \sqrt{\frac{c^4t^8}{16r^2} + c^2t^2}, \qquad (1\text{-}24)$$

$$= ct\sqrt{\frac{c^2t^6}{16r^2} + 1}. \qquad (1\text{-}25) \quad \textbf{[2 pt]}$$

Combining Eq. 1-18, Eq. 1-22, and Eq. 1-25, and then rearranging,

$$(\vec{\mathbf{v}} \cdot \vec{\mathbf{a}})^2 = v^2a^2\cos^2\phi, \qquad (1\text{-}26)$$

$$(vct)^2 = v^2(ct)^2\left(\frac{c^2t^6}{16r^2} + 1\right)\cos^2\phi, \qquad (1\text{-}27)$$

$$\frac{c^2t^6}{16r^2} = \frac{1}{\cos^2\phi} - 1, \qquad (1\text{-}28)$$

$$\frac{9s^2}{4r^2} = \frac{1}{\cos^2\phi} - 1, \qquad (1\text{-}29)$$

$$\frac{s}{r} = \frac{2}{3}\sqrt{\frac{1}{\cos^2\phi} - 1}, \qquad (1\text{-}30)$$

$$= \frac{2}{3}\tan\phi \qquad (1\text{-}31) \quad \textbf{[4 pt]}$$

Note that s/r is the angle, in radians, between P and Q.

It is stated that $\phi = 30^\circ$, so

$$\frac{s}{r} = \frac{2}{9}\sqrt{3}. \qquad (1\text{-}32) \quad \textbf{[4 pt]}$$

Question 2

(a) (13 points) The speed of the small block just before it collides with the large block, v_0, is found from conservation of mechanical energy:

$$\begin{aligned} K_i + U_i &= K_f + U_f, & (2\text{-}1) \\ \frac{1}{2}m{v_0}^2 + 0 &= 0 + mgy, & (2\text{-}2) \\ {v_0}^2 &= 2gy. & (2\text{-}3) \end{aligned}$$ **[3 pt]**

The speed of the large block just after the collision, v_2, is found from conservation of momentum,

$$\begin{aligned} 3mv_2 &= mv_0 + mv_1, & (2\text{-}4) \\ 3v_2 &= v_0 + v_1, & (2\text{-}5) \end{aligned}$$ **[2 pt]**

where v_1 is the speed of the small block just after the collision.

This collision is elastic, so kinetic energy is conserved, or

$$\begin{aligned} \frac{1}{2}3m{v_2}^2 &= \frac{1}{2}m{v_0}^2 - \frac{1}{2}m{v_1}^2, & (2\text{-}6) \\ 3{v_2}^2 &= {v_0}^2 - {v_1}^2. & (2\text{-}7) \end{aligned}$$ **[2 pt]**

Dividing Eq. 2-7 by Eq. 2-5 yields

$$v_2 = v_0 - v_1, \qquad (2\text{-}8)$$

which can be combined with Eq. 2-5 to give

$$\begin{aligned} v_1 &= v_0/2, & (2\text{-}9) \\ v_2 &= v_0/2 & (2\text{-}10) \end{aligned}$$ **[3 pt]**

The speed of the small block as it starts back up the path, v_1, is related to the height that it rises by conservation of mechanical energy (Eq. 2-1); this can be combined with Eq. 2-9 and Eq. 2-3 to show

$$\begin{aligned} K_i + U_i &= K_f + U_f, & (2\text{-}11) \\ 0 + mgh &= \frac{1}{2}m{v_1}^2 + 0, & (2\text{-}12) \\ gh &= \frac{1}{2}{v_1}^2, & (2\text{-}13) \\ &= \frac{1}{2}\left(\frac{v_0}{2}\right)^2, & (2\text{-}14) \\ &= \frac{1}{2}\frac{2gy}{4}, & (2\text{-}15) \\ h &= y/4. & (2\text{-}16) \end{aligned}$$ **[3 pt]**

(b) (12 points) Define v_3 to be the speed of the large block at the top of the loop. For the large block to just *barely* make it over the top of the loop requires the net force to be equal to the force of gravity.

$$F_{\text{net}} = 3mg = 3m\frac{v_3^2}{L} \qquad (2\text{-}17) \quad \textbf{[3 pt]}$$

The last expression comes from the condition that the tangential acceleration of the large block is zero at the top of the loop.

The speed of the large block at the top of the loop is related to the speed at the bottom by conservation of energy (Eq. 2-1).

$$\begin{aligned} K_t + U_t &= K_b + U_b \qquad &(2\text{-}18)\\ \frac{1}{2}3mv_3^2 + (3m)g(2L) &= \frac{1}{2}3mv_2^2 + 0, \qquad &(2\text{-}19)\\ v_3^2 + 4gL &= v_2^2. \qquad &(2\text{-}20) \quad \textbf{[3 pt]} \end{aligned}$$

Combining the last expression (Eq. 2-20) with the expression for net force (Eq. 2-17) yields

$$\begin{aligned} v_3^2 + 4gL &= v_2^2, \qquad &(2\text{-}21)\\ gL + 4gL &= v_2^2, \qquad &(2\text{-}22)\\ 5gL &= v_2^2. \qquad &(2\text{-}23) \quad \textbf{[3 pt]} \end{aligned}$$

The combine this result (Eq. 2-23) with the results of part (a), Eq. 2-10. Consequently

$$\begin{aligned} v_2 &= \frac{v_0}{2}, \qquad &(2\text{-}24)\\ v_2^2 &= \frac{v_0^2}{4}, \qquad &(2\text{-}25)\\ 5gL &= \frac{2gy}{4}, \qquad &(2\text{-}26)\\ L &= y/10. \qquad &(2\text{-}27) \quad \textbf{[3 pt]} \end{aligned}$$

Question 3

There are at least two ways to approach this problem. Other solution methods may exist as well.

Method 1

This approach focuses on the differential equations of oscillatory motion.

In general, any system satisfying an equation of the form

$$\ddot{z} = -\omega^2 z \qquad (3\text{-}1)$$

will oscillate with angular frequency ω. $\ddot{z}$ is shorthand for the second time derivative of the variable z, or

$$\ddot{z} \equiv \frac{d^2 z}{dt^2} \qquad (3\text{-}2)$$

Students don't need to write this down, but they will need to use it if they take this approach.

Let the rod have mass M and length $2r$. Let the springs each have constant k.

In experiment 1, let x be the vertical displacement of the rod from equilibrium. Then the equation of motion is

$$Ma = F_{net}, \tag{3-3}$$
$$M\ddot{x} = -2kx, \tag{3-4}$$
$$\ddot{x} = -\frac{2k}{M}x, \tag{3-5}$$
$$\omega_1 = \sqrt{\frac{2k}{M}}. \tag{3-6}$$ **[6 pt]**

where the 2 on the right hand side of Eq. 3-4 arises from the presence of two springs. In the last line we take advantage of Eq. 3-1: it does *not* need to be derived. Note the *absence* of gravity from this expression. x is the change in length from the equilibrium position, and then net force at the equilibrium position is zero.

In experiment 2, let θ be the angular displacement of the rod from equilibrium. Note that for small θ, the extension of each spring from equilibrium is given by

$$x = r\theta. \tag{3-7}$$ **[2 pt]**

The force of the spring on each end of the rod is

$$F = -kx = -kr\theta, \tag{3-8}$$ **[2 pt]**

so the net torque on the rod about the center of rotation is

$$\tau_{net} = 2(kr\theta)r. \tag{3-9}$$ **[2 pt]**

The rotational inertia of the rod of length $2r$ about the center is

$$I = \frac{1}{12}M(2r)^2 = \frac{1}{3}Mr^2. \tag{3-10}$$ **[3 pt]**

Then, combining Eq. 3-7 through Eq. 3-10,

$$I\alpha = \tau \tag{3-11}$$
$$\frac{1}{3}Mr^2\ddot{\theta} = -2kr^2\theta, \tag{3-12}$$
$$\ddot{\theta} = \frac{6k}{M}\theta, \tag{3-13}$$
$$\omega_2 = \sqrt{\frac{6k}{M}}. \tag{3-14}$$ **[6 pt]**

Finally, since

$$T = \frac{2\pi}{\omega}, \quad (3\text{-}15) \quad \textbf{[2 pt]}$$

we have from Eq. 3-6 and Eq. 3-14

$$\frac{T_1}{T_2} = \frac{\omega_2}{\omega_1}, \quad (3\text{-}16)$$

$$= \sqrt{3} \quad (3\text{-}17) \quad \textbf{[2 pt]}$$

Method 2

This method focuses on the fundamental expression for the period of an oscillating system.

The period of oscillation of a mass spring system is given by

$$T = 2\pi\sqrt{\frac{m}{k}}, \quad (3\text{-}18) \quad \textbf{[3 pt]}$$

where k is the spring constant and m is the mass.

In the first experiment there are two springs, each effectively moving *half* of the total mass M of the thin rod, so

$$T_1 = 2\pi\sqrt{\frac{M/2}{k}} = 2\pi\sqrt{\frac{M}{2k}}. \quad (3\text{-}19) \quad \textbf{[2 pt]}$$

The period of oscillation of a rotational system is given by

$$T_2 = 2\pi\sqrt{\frac{I}{\kappa}} \quad (3\text{-}20) \quad \textbf{[3 pt]}$$

where κ is the rotational spring constant and I the rotational inertia. In this case,

$$\kappa\theta = \tau \quad (3\text{-}21) \quad \textbf{[2 pt]}$$

where τ is the torque about the center of rotation. That torque is provided for by the force of the spring,

$$\tau = rF = \frac{L}{2}kx. \quad (3\text{-}22) \quad \textbf{[4 pt]}$$

where x is the extension of the spring and L the length of the thin rod.

The angle θ is given by

$$\theta \approx \frac{x}{L/2}, \quad (3\text{-}23) \quad \textbf{[2 pt]}$$

for small x, so

$$\kappa = \tau/\theta, \quad (3\text{-}24)$$
$$= k\frac{L^2}{4}. \quad (3\text{-}25) \quad \textbf{[2 pt]}$$

for *one* spring.

The rotational inertia for a thin rod about the center is

$$I_{\text{rod}} = \frac{1}{12}ML^2, \quad (3\text{-}26) \quad \textbf{[1 pt]}$$

but we are interested in only *half*, so

$$I = \frac{1}{24}ML^2 \quad (3\text{-}27) \quad \textbf{[2 pt]}$$

Combine Eq. 3-20, Eq 3-25, and Eq. 3-27. The period of oscillation is then

$$T_2 = 2\pi\sqrt{\frac{I}{\kappa}}, \quad (3\text{-}28)$$
$$= 2\pi\sqrt{\frac{ML^2/24}{kL^2/4}}, \quad (3\text{-}29)$$
$$= 2\pi\sqrt{\frac{M}{6k}}. \quad (3\text{-}30) \quad \textbf{[2 pt]}$$

The ratio is found by dividing Eq. 3-19 by Eq. 3-30:

$$\frac{T_1}{T_2} = \sqrt{3}. \quad (3\text{-}31) \quad \textbf{[2 pt]}$$

Question 4

(a) (13 points) The magnitude of the net force on an object of mass m in uniform circular motion of radius r is

$$F_{\text{net}} = m\frac{v^2}{r} \quad (4\text{-}1) \quad \textbf{[1 pt]}$$

but

$$v = 2\pi r/T, \quad (4\text{-}2) \quad \textbf{[1 pt]}$$

where T is the period of revolution of the object. Combining the previous two expressions yields

$$F_{\text{net}} = m\frac{4\pi^2 r}{T^2} \quad (4\text{-}3) \quad \textbf{[1 pt]}$$

Both the particle and the asteroid have the same period of revolution. In the case of the asteroid the net force is equal to the gravitational attraction between the planet and the asteroid, so

$$F_{\text{net}} = M_a \frac{4\pi^2 r}{T^2} = \frac{G M_a M_p}{r^2} \tag{4-4}$$

or

$$T = \sqrt{4\pi^2 \frac{r^3}{G M_p}} \tag{4-5}$$ [2 pt]

The distance from the particle of mass m to the center of rotation is

$$r_m = r - R_a \tag{4-6}$$

The magnitude of the net force on the particle of mass m is

$$F_{\text{net}} = m \frac{4\pi^2 r_m}{T^2} \tag{4-7}$$ [1 pt]

which can be combined with Eq 4-5 for the period of the asteroid to yield

$$F_{\text{net}} = G m M_p \frac{r_m}{r^3}. \tag{4-8}$$ [2 pt]

The magnitude of the force of gravity from the planet on the particle of mass m is

$$F_p = \frac{G m M_p}{{r_m}^2} \tag{4-9}$$ [1 pt]

The magnitude of the force of gravity from the asteroid on the particle of mass m is

$$F_a = \frac{G m M_a}{{R_a}^2} \tag{4-10}$$ [1 pt]

The magnitude of the net force on the particle can also be written as

$$F_{\text{net}} = F_p - F_N - F_a. \tag{4-11}$$ [1 pt]

where F_N is the normal force, so

$$\begin{aligned} F_N &= F_{\text{net}} + F_a - F_p, &(4\text{-}12)\\ &= Gm\left(M_p \frac{r_m}{r^3} + M_a \frac{1}{{R_a}^2} - M_p \frac{1}{{r_m}^2}\right), &(4\text{-}13)\\ &= Gm\left(M_p \frac{r - R_a}{r^3} + M_a \frac{1}{{R_a}^2} - M_p \frac{1}{(r - R_a)^2}\right), &(4\text{-}14) \end{aligned}$$ [2 pt]

(b) (7 points) Rewrite the terms, where possible, so that

$$r - R_a = r(1 - x), \quad (4\text{-}15)$$

where

$$x = R_a/r \quad (4\text{-}16)$$

is a small quantity. Then Eq. 4-14 can be written as

$$F_N = Gm\left(M_p\frac{1-x}{r^2} + M_a\frac{1}{{R_a}^2} - M_p\frac{1}{r^2(1-x)^2}\right). \quad (4\text{-}17) \quad \textbf{[2 pt]}$$

Apply a binomial expansion to the last term,

$$(1-x)^{-2} \approx 1 + 2x, \quad (4\text{-}18) \quad \textbf{[2 pt]}$$

then Eq. 4-14 will be approximated by

$$F_N \approx Gm\left(M_p\frac{1-x}{r^2} + M_a\frac{1}{{R_a}^2} - M_p\frac{1}{r^2}(1+2x)\right), \quad (4\text{-}19)$$

$$= Gm\left(M_a\frac{1}{{R_a}^2} - 3M_p\frac{1}{r^2}x\right), \quad (4\text{-}20)$$

$$= Gm\left(M_a\frac{1}{{R_a}^2} - 3M_p\frac{R_a}{r^3}\right). \quad (4\text{-}21) \quad \textbf{[3 pt]}$$

(c) (5 points) Set the normal force equal to zero,

$$F_N = 0, \quad (4\text{-}22) \quad \textbf{[1 pt]}$$

and solve Eq. 4-21 for r,

$$r = \sqrt[3]{3\frac{M_p}{M_a}}\,R_a. \quad (4\text{-}23) \quad \textbf{[1 pt]}$$

The mass of the planet is equal to the density ρ times the volume, so

$$M_p = \frac{4}{3}\pi\rho {R_p}^3. \quad (4\text{-}24) \quad \textbf{[1 pt]}$$

A similar expression holds true for the asteroid, so the mass ratio is

$$\frac{M_p}{M_a} = \frac{{R_p}^3}{{R_a}^3} \quad (4\text{-}25) \quad \textbf{[1 pt]}$$

Combining Eq. 4-23 and Eq. 4-25,

$$r = \sqrt[3]{3}\,R_p \quad (4\text{-}26) \quad \textbf{[1 pt]}$$