2008 Quarter-Final Exam Solutions

1. A charged particle with charge q and mass m starts with an initial kinetic energy K at the middle of a uniformly charged spherical region of total charge Q and radius R. q and Q have opposite signs. The spherically charged region is not free to move.

   (a) Find the value of K_0 such that the particle will just reach the boundary of the spherically charged region.

   (b) How much time does it take for the particle to reach the boundary of the region if it starts with the kinetic energy K_0 found in part (a)?

   Solution:

   Assume that q is negative and Q is positive.

   (a) Apply Gauss's Law to a spherical shell of radius r where $r < R$. Then,

$$E 4\pi r^2 = \frac{4\pi\rho r^3}{\epsilon_0}$$

   where the charge density $\rho = \frac{3Q}{4\pi R^3}$

   Solving for the Electric Field at a distance r from the center, we find

$$E = \frac{\rho r}{3\epsilon_0}$$

   We now find the potential difference between the center of the sphere and the outer boundary of the charged cloud.

$$\Delta V = -\int_0^R E dr$$

   Substituting in our expression for E, we now find

$$\Delta V = -\frac{\rho R^2}{6\epsilon_0}$$

   Using conservation of energy,

$$K_{lost} = U_{gained}$$

$$K_0 = -q\Delta V = \frac{-qQ}{8\pi R\epsilon_0}$$

   (b) Apply Newton's Second Law to the object of charge $-q$ when it is located a distance r away from the center of the sphere.

$$F_{net} = ma$$

$$-(-q)E = m\frac{d^2r}{dt^2}$$

$$\frac{d^2r}{dt^2} = -\frac{-q\rho r}{3\epsilon_0 m}$$

We recognize that this is the differential equation for simple harmonic motion

$$\frac{d^2r}{dt^2} = -\omega^2 r$$

where $\omega = \sqrt{\frac{-q\rho}{3\epsilon_0 m}}$.

Since the charge has the minimum kinetic energy needed to reach the surface, the trip from the center of the sphere to the outer boundary is one-fourth of a cycle of SHM.

Therefore,

$$t = \frac{T}{4} = \frac{\pi}{2\omega} = \frac{\pi}{2}\sqrt{\frac{3\epsilon_0 m}{-q\rho}}$$

.

Substituting, in for rho, we find

$$t = \frac{\pi}{2}\sqrt{\frac{4\pi\epsilon_0 m R^3}{-qQ}}$$

.

2. A uniform pool ball of radius r begins at rest on a pool table. The ball is given a horizontal impulse J of fixed magnitude at a distance βr above its center, where $-1 \le \beta \le 1$. The coefficient of kinetic friction between the ball and the pool table is μ. You may assume the ball and the table are perfectly rigid. Ignore effects due to deformation. (The moment of inertia about the center of mass of a solid sphere of mass m and radius r is $I_{cm} = \frac{2}{5}mr^2$.)

   (a) Find an expression for the final speed of the ball as a function of J, m, and β.

   (b) For what value of β does the ball immediately begin to roll without slipping, regardless of the value of μ.

.

Solution 1:

(a) Consider an axis perpendicular to the initial impulse and coplanar with the table. (Throughout this solution we consider only torques and angular momenta with respect to this axis.) After the initial impulse, the torque about this axis is always zero, so angular momentum is conserved. The initial impulse occurs a perpendicular distance $(\beta + 1)\, r$ from the axis, so the angular momentum is

$$L = (\beta + 1)\, rJ$$

After the ball has skidded along the table for a certain distance, it will begin to roll without slipping. At that point, $\omega = \frac{v_f}{r}$.

Meanwhile, its moment of inertia about this axis is (by the parallel-axis theorem) $I = I_{cm} + mr^2 = \frac{7}{5}mr^2$, so that its final angular velocity ω is given by

$$L = I\omega$$

$$(\beta + 1)\, rJ = \frac{7}{5}mr^2\omega$$

$$(\beta + 1)\, J = \frac{7}{5}mv_f$$

.

Therefore,

$$v_f = \frac{5J}{7m}(1 + \beta)$$

.

(b) The ball acquires linear momentum J as a result of the horizontal impulse, so its initial velocity v is given by

$$mv = J$$

We want the initial angular velocity and initial velocity to satisfy the no-slip condition $v = \omega r$; thus

$$(\beta + 1)\, J = \frac{7}{5}J$$

$$\beta = \frac{2}{5}$$

.

Solution 2:.

Consider torques and angular momenta about the center of mass. If the horizontal impulse is large compared with the horizontal impulse from friction during the time that the cue stick is in contact with the ball, then angular impulse = change in angular momentum becomes:

$$J\beta r = \frac{2}{5}mr^2\omega_0$$

.

Linear impulse = change in linear momentum yields:

$$J = mv_0$$

.

(b) We want the initial angular velocity and initial velocity to satisfy the no-slip condition $v_0 = \omega_0 r$; thus

$$J\beta = \frac{2}{5}mv_0 = \frac{2}{5}J$$

.

$$\beta = \frac{2}{5}$$

.

(a) In the case that the ball does not immediately begin to roll without slipping, friction will exert an angular impulse, ftr, about the center of mass of the ball as the ball skids along the surface:

$$ftr = \frac{2}{5}mr^2(\omega_f - \omega_0)$$

.

Friction will also exert a linear impulse $-ft$ that will cause a change in linear momentum:

$$-ft = m(v_f - v_0)$$

.

Combining the last two equations:

$$-m(v_f - v_0) = \frac{2}{5}mr(\omega_f - \omega_0)$$

After the ball has skidded along the table for a certain distance, it will begin to roll without slipping. At that point, $\omega = \frac{v_f}{r}$.

$$-v_f + v_0 = \frac{2}{5}v_f - \frac{2}{5}r\omega_0$$

Now, using the relationships between the impulse J and the initial angular and linear momentum,

$$-v_f + \frac{J}{m} = \frac{2}{5}v_f - \frac{J\beta}{m}$$

.

$$\frac{J}{m}(1+\beta) = \frac{7}{5}v_f$$

.

$$v_f = \frac{5J}{7m}(1+\beta)$$

.

3. A block of mass m slides on a circular track of radius r whose wall and floor both have coefficient of kinetic friction μ with the block. The floor lies in a horizontal plane and the wall is vertical. The block is in constant contact with both the wall and the floor. The block has initial speed v_0.

   (a) Let the block have kinetic energy E after traveling through an angle θ. Derive an expression for $\frac{dE}{d\theta}$ in terms of g, r, μ, m and E.

   (b) Suppose the block circles the track exactly once before coming to a halt. Determine v_0 in terms of g, r, and μ.

Solution Let v be the speed of the block after it has traveled through an angle θ. When we apply Newton's Second Law to the block as it is traveling around the circular path of radius r at a speed v, we find that the force of the wall on the block, F_w, is

$$F_w = \frac{mv^2}{r}$$

and the force of the floor on the block, F_N, is

$$F_N = mg$$

Therefore, the total force of friction on the block is

$$F_{Fric} = \mu m(\frac{v^2}{r} + g)$$

.

Let dE denote the loss of kinetic energy as a result of the work W that friction does on the block when it has traveled through an angle $d\theta$. The block will travel a distance $ds = rd\theta$ as it travels through an angle $d\theta$.

Then,

$$dE = W = -F_{Fric}ds = -F_{Fric}rd\theta$$

.

Now, substituting in the expressions for F_{Fric}, we find

$$dE = -\mu m(v^2 + gr)d\theta$$

.

$$\frac{dE}{d\theta} = -\mu(mv^2 + mgr) = -\mu(2E + mgr)$$

where we have simply used the fact that the kinetic energy E is $E = \frac{mv^2}{2}$.

(b) Now, we separate variables and solve the differential equation.

$$\int \frac{dE}{2E + mgr} = \int -\mu d\theta$$

.

$$\frac{1}{2}\ln|2E + mgr| = -\mu\theta + C$$

.

Exponentiating both sides, yields

$$2E + mgr = Ce^{-2\mu\theta}$$

Let E_0 denote the initial kinetic energy. Then, using initial conditions, we find that $C = 2E_0 + mgr$.

Now, using the fact that $E = 0$ when $\theta = 2\pi$, we obtain

$$mgr = (2E_0 + mgr)e^{-4\pi\mu}$$

.

Solving for E_0,

$$E_0 = \frac{1}{2}mgr(e^{4\mu\pi} - 1)$$

.

Then, using $E_0 = \frac{mv_0^2}{2}$, and we find that

$$v_0 = \sqrt{gr(e^{4\mu\pi} - 1)}$$

.

4. Two beads, each of mass m, are free to slide on a rigid, vertical hoop of mass m_h. The beads are threaded on the hoop so that they cannot fall off of the hoop. They are released with negligible velocity at the top of the hoop and slide down to the bottom in opposite directions. What is the maximum value of the ratio m/m_h such that the hoop always remains in contact with the ground? Neglect friction.

Solution 1: Draw a free-body diagram for each bead; let F_N be the (inward) normal force exerted by the hoop on the bead. Let θ be the angular position of the bead, measured from the top of the hoop, and let the hoop have radius r. We see that

$$F_N + mg\cos\theta = m\frac{v^2}{r}$$

$$F_N = m\frac{v^2}{r} - mg\cos\theta$$

The (downward) vertical component F_{Ny} is given by

$$F_{Ny} = F_N\cos\theta$$

From Newton's third law, the two beads together exert an upwards vertical force on the hoop given by

$$F_u = 2F_{Ny}$$

$$F_u = 2m\cos\theta\left(\frac{v^2}{r} - g\cos\theta\right)$$

noting that the beads clearly reach the same position at the same time.

Meanwhile, when each bead is at a position θ it has moved through a vertical distance $r(1-\cos\theta)$. Thus from energy conservation,

$$\frac{1}{2}mv^2 = mgr(1-\cos\theta)$$

$$\frac{v^2}{r} = 2g(1-\cos\theta)$$

Inserting this into the previous result,

$$F_u = 2m\cos\theta\,(2g(1-\cos\theta) - g\cos\theta)$$

$$F_u = 2mg\left(2\cos\theta - 3\cos^2\theta\right)$$

If the beads ever exert an upward force on the hoop greater than $m_h g$, the hoop will leave the ground; *i.e.*, the condition for the hoop to remain in contact with the ground is that for all θ,

$$F_u \le m_h g$$

We can replace the left side by its maximum value. Letting $s = \cos\theta$,

$$F_u = 2mg\left(2s - 3s^2\right)$$

$$\frac{d}{ds}F_u = 2mg\left(2 - 6s\right)$$

The derivative is zero at $s = \frac{1}{3}$, where

$$F_{u(max)} = \frac{2}{3}mg$$

So our condition is

$$\frac{2}{3}mg \le m_h g$$

$$\frac{m}{m_h} \le \frac{3}{2}$$

Solution 2: As before, we apply energy conservation to find the speed of the beads:

$$v = \sqrt{2gr\left(1 - \cos\theta\right)}$$

The vertical (downward) component of the beads' velocity is thus

$$v_y = v\sin\theta$$

$$v_y = \sin\theta\sqrt{2gr\left(1 - \cos\theta\right)}$$

While the hoop is in contact with the ground, the beads are the only part of the system in motion, so the momentum of the system is simply the beads' momentum:

$$p_y = 2mv_y$$

$$p_y = 2m\sin\theta\sqrt{2gr\left(1 - \cos\theta\right)}$$

The net (downward) force on the system is

$$F_{net} = \frac{dp_y}{dt} = \frac{dp_y}{d\theta}\frac{d\theta}{dt} = \frac{dp_y}{d\theta}\frac{v}{r}$$

$$F_{net} = 2m\sqrt{2gr}\frac{d}{dt}\left(\sin\theta\sqrt{1 - \cos\theta}\right)\cdot\frac{1}{r}\sqrt{2gr\left(1 - \cos\theta\right)}$$

$$F_{net} = 4mg\left(\cos\theta\sqrt{1 - \cos\theta} + \sin\theta\frac{1}{2\sqrt{1 - \cos\theta}}\sin\theta\right)\sqrt{1 - \cos\theta}$$

$$F_{net} = 4mg\left(\cos\theta - \cos^2\theta + \frac{1}{2}\sin^2\theta\right)$$

$$F_{net} = 2mg\left(2\cos\theta - 3\cos^2\theta + 1\right)$$

This downward force is provided by gravity and the normal force upward on the hoop:

$$F_{net} = 2mg + m_h g - F_N$$

If the hoop is to remain in contact with the ground, the normal force can never be negative:

$$F_N \geq 0$$

$$2mg + m_h g - F_{net} \geq 0$$
$$2mg + m_h g - 2mg\left(2\cos\theta - 3\cos^2\theta + 1\right) \geq 0$$
$$2mg\left(2\cos\theta - 3\cos^2\theta\right) \leq m_h g$$

and we proceed as above.