AAPT UNITED STATES PHYSICS TEAM
AIP 2023

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination has two parts. Each part has three questions and lasts for 90 minutes.
- For each student, print out one copy of the exam and one copy of the answer sheets. Print everything single-sided, and do not staple anything. Divide the exam into the instructions (pages 2–3), Part A questions (pages 4–8), and Part B questions (pages 9–11).
- Begin by giving students the instructions and all of the answer sheets. Let the students read the instructions and fill out their information on the answer sheets. They can keep the instructions for both parts of the exam. Also give students blank sheets of paper to use as scratch paper throughout the exam.
- Students may bring calculators, but they may not use symbolic math, programming, or graphing features of these calculators. Calculators may not be shared, and their memory must be cleared of data and programs. Cell phones or other electronics may not be used during the exam or while the exam papers are present. Students may not use books or other references.
- To start the exam, give students the Part A questions, and allow 90 minutes to complete Part A. Do not give students Part B during this time, even if they finish with time remaining. At the end of the 90 minutes, collect the Part A questions and answer sheets.
- Then give students a 5 to 10 minute break. Then give them the Part B questions, and allow 90 minutes to complete Part B. Do not let students go back to Part A.
- At the end of the exam, collect everything, including the questions, the instructions, the answer sheets, and the scratch paper. Students may *not* keep the exam questions. Everything can be returned to the students after April 19th, 2023.
- After the exam, sort each student's answer sheets by page number. Scan every answer sheet, including blank ones.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

Tengiz Bibilashvili, Kellan Colburn, Samuel Gebretsadkan, Abi Krishnan, Natalie LeBaron, Kye Shi, Brian Skinner, and Kevin Zhou.

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AIP 2023

USA Physics Olympiad Exam

Instructions for the Student

- You should receive these instructions, the reference table on the next page, answer sheets, and blank paper for scratch work. Read this page carefully before the exam begins.
- You may use a calculator, but its memory must be cleared of data and programs, and you may not use symbolic math, programming, or graphing features. Calculators may not be shared. Cell phones or other electronics may not be used during the exam or while the exam papers are present. You may not use books or other outside references.
- When the exam begins, your proctor will give you the questions for Part A. You will have 90 minutes to complete three problems. Each question is worth 25 points, but they are not necessarily of the same difficulty. If you finish all of the questions, you may check your work, but you may not look at Part B during this time.
- After 90 minutes, your proctor will collect the questions and answer sheets for Part A. You may then take a short break.
- Then you will work on Part B. You have 90 minutes to complete three problems. Each question is worth 25 points. Do not look at Part A during this time. When the exam ends, you must return all papers to the proctor, including the exam questions.
- **Do not discuss the questions of this exam, or their solutions, until after April 19th, 2023. Violations of this rule may result in disqualification.**

Below are instructions for writing your solutions.

- All of your solutions must be written on the official answer sheets. Nothing outside these answer sheets will be graded. Before the exam begins, write your name, student AAPT number, and proctor AAPT number as directed on the answer sheets.
- There are several answer sheets per problem. If you run out of space for a problem, you may use the extra answer sheets, which are at the end of the answer sheet packet. To ensure this work is graded, you must indicate, at the bottom of your last answer sheet for that problem, that you are using these extra answer sheets.
- Only write within the frame of each answer sheet. To simplify grading, we recommend drawing a $\boxed{\text{box}}$ around your final answer for each subpart. You should organize your work linearly and briefly explain your reasoning, which will help you earn partial credit. You may use either pencil or pen, but sure to write clearly so your work will be legible after scanning.

Reference table of possibly useful information

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9$ N $\cdot$ m^2/C^2	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}$ T $\cdot$ m/A
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}$ (mol)$^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31$ J/(mol $\cdot$ K)
$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)	$e = 1.602 \times 10^{-19}$ C
1 eV $= 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s
$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \theta^3/6$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \theta^2/2$ for $\lvert\theta\rvert \ll 1$

Possibly useful integrals

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C \qquad \int \frac{dx}{1-x^2} = \tanh^{-1}(x) + C \qquad \int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + C \qquad \int \frac{dx}{1+x^2} = \tan^{-1}(x) + C \qquad \int \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{\sqrt{1+x^2}} + C$$

$$\int \sqrt{\frac{1+x}{1-x}}\, dx = -\sqrt{1-x^2} - 2\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right) + C$$

$$\int \frac{\sqrt{1+x}}{(1-x)^{3/2}}\, dx = 2\sqrt{\frac{1+x}{1-x}} + 2\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right) + C$$

$$\int \frac{dx}{(1-x)^{3/2}(1+x)^{1/2}} = \sqrt{\frac{1+x}{1-x}} + C$$

$$\int \frac{dx}{(1-x)^{3/2}(1+x)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$$

$$\int \frac{dx}{(1-x)^{5/2}(1+x)^{1/2}} = \frac{(2-x)\sqrt{1+x}}{3(1-x)^{3/2}} + C$$

$$\int \frac{dx}{(1-x)^{5/2}(1+x)^{3/2}} = \frac{1+2x-2x^2}{3(1-x)^{3/2}(1+x)^{1/2}} + C$$

You may use this sheet for both parts of the exam.

End of Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Part A

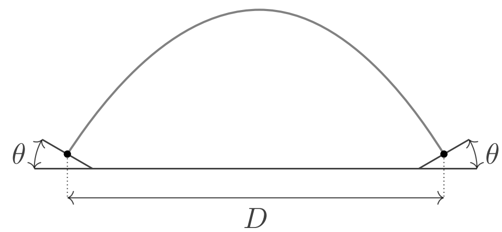
Question A1

Circus Act

In this problem we consider a small ball bouncing back and forth between two points. In all parts below, the acceleration of gravity is g, collisions are perfectly elastic, air resistance is negligible, and the impact points are at the same height. The diagrams are not drawn to scale.

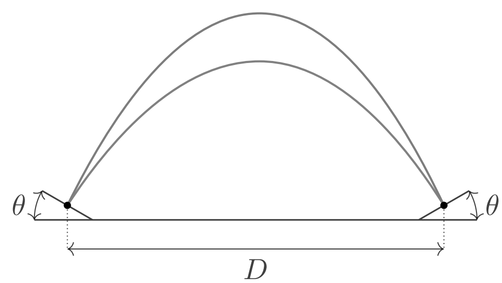
a. Consider a ball bouncing between two inclined planes, which each make an angle $\theta < 90°$ to the horizontal. The ball has speed v_0 at the impact points, which are separated by a distance D.

   i. The ball can bounce back and forth along the same path, as shown.



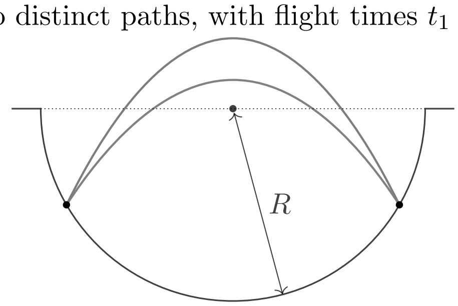
   For what values of θ is this motion possible? For these values, what is v_0?

   ii. The ball can also take one trajectory while traveling to the right, and a separate trajectory when traveling back. Let $\phi \neq 0$ be the angle between the paths at the impact points.



   For what values of θ and ϕ is this motion possible? For these values, what is v_0?

b. Now suppose the ball bounces within a hemispherical well of radius of curvature R. As in part a.ii, it alternates between two distinct paths, with flight times t_1 and $t_2 \neq t_1$.



   Find all of the possible values of R, in terms of t_1 and t_2.

c. Finally, suppose the well has a sinusoidal shape, described by $y(x) = -L\sin\big(2x/L\big)$. The ball takes two distinct paths with flight times t_1 and $t_2 \neq t_1$, and the horizontal distance between the impact points is less than πL. Find all of the possible values of L, in terms of t_1 and t_2.

Question A2

Time is a Flat Circle

A particle of mass m and negative charge $-q$ is constrained to move in a horizontal plane. In the situations described below, this particle can either oscillate back and forth in a straight line or move in a circle. (These two modes of motion are interesting because they generate linearly and circularly polarized radiation, respectively, but in this problem you may ignore any energy lost to radiation.)

a. A large positive charge $Q \gg q$ is fixed in place a distance R directly below the origin of the plane.

   i. When the particle is a distance $r \ll R$ from the origin, find an approximate expression for its potential energy due to the charge Q to second order in r/R, up to an arbitrary constant. You may use this result for the rest of the problem.

   ii. If the particle oscillates linearly with amplitude $a \ll R$, what is its angular frequency ω_ℓ?

   iii. If the particle performs circular motion with radius $r \ll R$, what is its angular frequency ω_c?

b. Now an additional negative charge $-q$ is fixed in place at the origin of the plane.

   i. What is the equilibrium distance r_0 of the particle from the origin?

   ii. If the particle oscillates linearly with amplitude $a \ll r_0$, what is its angular frequency Ω_ℓ? Is it higher or lower than ω_ℓ?

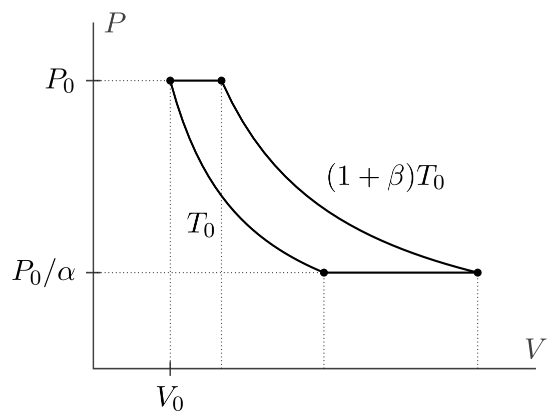
   iii. Now suppose the particle performs circular motion with radius $r = r_0 + \delta r$, where $\delta r \ll r_0$. What is its angular frequency Ω_c, in terms of ω_c, r, and δr? Is it higher or lower than ω_c?

Question A3

The Motive Power of Ice

In the Carnot cycle, a gas is heated at constant temperature T_H and cooled at constant temperature T_C. Furthermore, no other heat transfer occurs, and all other steps of the cycle are reversible. The laws of thermodynamics state that any such cycle must have efficiency $\eta = W/Q_{\text{in}} = 1 - (T_C/T_H)$. Below we will explore two other heat engines, which recover this efficiency in certain limits.

a. Consider the following heat engine involving one mole of ideal monatomic gas. The gas begins at temperature T_0, pressure P_0, and volume V_0, and undergoes four reversible steps.



1. The gas is expanded at constant pressure until its temperature rises to $(1+\beta)T_0$.
2. The gas is expanded at constant temperature until its pressure falls to P_0/α.
3. The gas is contracted at constant pressure until its temperature falls back to T_0.
4. The gas is contracted at constant temperature until its pressure rises back to P_0.

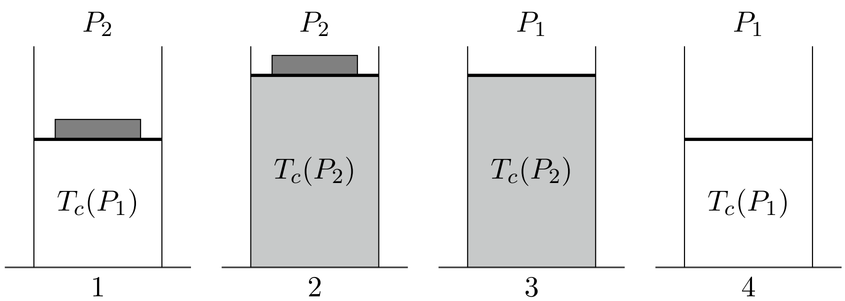
i. Which steps require heat to be transferred to the gas? For each such step, give the total heat input in terms of P_0, V_0, α, and β.

ii. Under what conditions on α and β would we expect the efficiency of this heat engine to approach that of a Carnot cycle working between the same maximum and minimum temperatures?

iii. Find the efficiency of this heat engine for general α and β.

The second half of the problem is on the next page.

b. Now consider a heat engine built around the freezing and melting of water, which occurs at a pressure-dependent temperature $T_c(P)$. Initially, a volume of V of water is squeezed underneath a piston, so that it experiences a total pressure P_1, and the water is on the edge of freezing, with temperature $T_c(P_1)$. The engine then undergoes four reversible steps.



1. A mass is slowly placed on the piston, raising the total pressure to P_2.
2. The water is cooled to temperature $T_c(P_2)$ and frozen.
3. The mass is slowly removed from the piston, lowering the pressure back to P_1.
4. The ice is heated back to temperature $T_c(P_1)$ and melted.

Assume that water and ice are incompressible, with fixed densities ρ_w and ρ_i.

i. What is the net work done by this engine, in terms of P_1, P_2, V, and the densities?

ii. Assume the latent heat per unit mass L to melt ice is large, so that freezing and melting account for essentially all of the heat transfer in the cycle. What is the efficiency of the engine, in terms of P_1, P_2, L, and the densities?

iii. Since we assumed all heat transfer occurs during melting or freezing, this cycle has the same efficiency as a Carnot cycle. In the limit where P_1 and P_2 are very close, use this fact to infer an expression for dT_c/dP in terms of T_c, L, and the densities.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you can review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

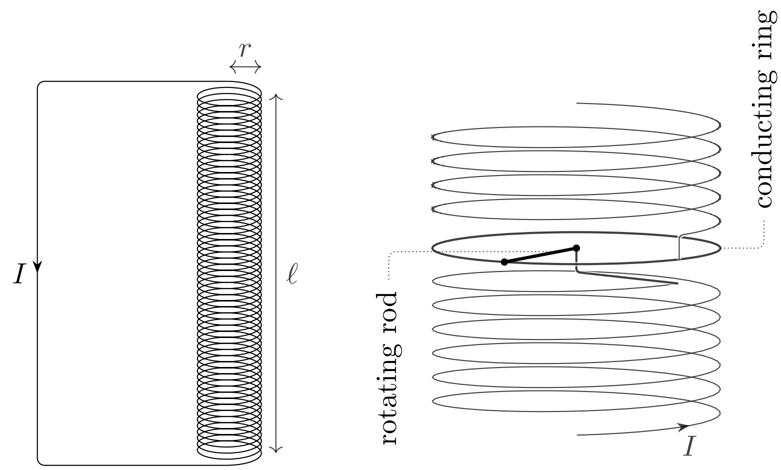
Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

Electric Roulette

Consider a cylindrical solenoid with radius r, length $\ell \gg r$, and n turns per unit length. It is made of one continuous wire, with the top connecting back to the bottom as shown at left.



In the middle of the solenoid, part of the wire is replaced with the assembly shown at right. A uniform conducting rod of mass m and radius r is connected to the bottom half of the solenoid, and is free to rotate about the solenoid's axis of symmetry. The end of the rod slides on a fixed conducting ring, which is attached to the top half of the solenoid. This assembly and the solenoid form one continuous conductor, carrying total current I.

a. What is the inductance of this system? Assume $nr \gg 1$, so that the magnetic field produced by the current in the rod and ring is negligible.

b. When the rod is within a uniform vertical magnetic field B, find the torque it experiences in terms of I, B, and r.

c. If the rod rotates with angular velocity ω, the electrons inside have a tangential velocity. Find the electromotive force across the rod in terms of ω, B, and r.

Now we will consider the dynamics of this system in some simple situations. For all the parts below, neglect energy losses due to friction, resistance, and radiation.

d. First, suppose the system initially carries no current, and the entire system is inside a uniform external magnetic field B_0 parallel to the axis of the solenoid. If the rod is given a small initial angular velocity, its angular velocity will oscillate in time. Find the period of these oscillations.

e. Next, suppose there is no external magnetic field, $B_0 = 0$, and at time $t = 0$, the system carries current I_0 and the rod has zero angular velocity.

   i. The rod's angular velocity $\omega(t)$ approaches a value ω_0 after a long time. What is ω_0?

   ii. Find $\omega(t)/\omega_0$ in terms of ω_0, t, n, and ℓ. You may use the integrals on the reference sheet.

Question B2

Fast and Furious

A space program wants to accelerate a spaceship of final mass $m = 100\,\text{kg}$ to relativistic speeds to observe distant stars. They have two proposals to evaluate.

a. Their first proposal is to use traditional rocket propulsion. A rocket of initial mass m_0 and final mass m that expels propellant with exhaust speed u relative to the rocket will reach a speed

$$v = u \ln\left(\frac{m_0}{m}\right).$$

Suppose the desired final speed is $v_f = 3c/5$. In the subparts below, neglect relativistic effects and give your answers in the form 10^n, where n has at least two significant figures.

i. If the rocket has exhaust speed $u = 3.5\,\text{km/s}$, what must its starting mass be in kilograms?

ii. If the propellant is exhausted at rate $7.0\,\text{kg/s}$, how long does the acceleration take, in centuries?

iii. If the energy density of the fuel is $2.0 \times 10^7\ \text{J/kg}$, how much total energy is required, in Joules?

For the rest of this problem, you should account for special relativity.

b. Another option is to use a spaceship with constant mass m, propelled by light produced by lasers on Earth, with total power $P = 6 \times 10^{12}\,\text{W}$. The light evenly impacts a sail on the spaceship, and reflects off the sail directly back towards the Earth. Neglect the orbital motion of the Earth, and give all your answers in the frame of the Earth.

i. What is the force on the spaceship when the spaceship has speed v?

ii. How long will it take to accelerate the spaceship to speed $v_f = 3c/5$, in seconds? You may use the integrals on the reference sheet.

iii. At the moment the spaceship reaches this speed, how much total energy has been used to power the lasers, in Joules?

The following results from relativity may be helpful:

- The Lorentz factor is defined as $\gamma = 1/\sqrt{1 - v^2/c^2}$.
- An object of mass m and velocity $\mathbf{v}$ has momentum $\mathbf{p} = \gamma m \mathbf{v}$ and energy $E = \gamma mc^2$. The force is defined by $\mathbf{F} = d\mathbf{p}/dt$.
- The momentum and energy of light are related by $E = pc$.
- In a frame S' with velocity $v\hat{\mathbf{x}}$ relative to a frame S, the energy and momentum are

$$E' = \gamma(E - vp_x), \quad p'_x = \gamma(p_x - vE/c^2).$$

Question B3

Starry Messengers

In 1987, light from supernova SN1987A was detected by telescopes on Earth. The supernova occurred in the Large Magellanic Cloud, a distance $d = 1.5 \times 10^{21}$ m away, making it the closest in centuries. Observations of this event tell us a remarkable amount about elementary particles.

a. Both light and neutrinos were produced in the core of the supernova. Neutrinos are elementary particles which interact extremely weakly with ordinary matter. Detectors on Earth saw a few dozen of these neutrinos, in a burst which occurred about $T = 3\,\text{hours}$ *before* the light arrived.

   i. One explanation of these observations is that the neutrinos' speed v was faster than the speed of light c, violating special relativity. If this is the case, find $v - c$ in m/s.

   ii. Another explanation is that the light was slowed down by the gas in the solar system, while the neutrinos always moved at speed c. Suppose the solar system has a uniform index of refraction n within a radius $D = 10^{13}$ m. What would n have to be to explain the time delay?

   Neither of these explanations seem plausible; the modern accepted explanation is that the light was trapped for some time inside the supernova, while the neutrinos were able to leave immediately. Therefore, for the rest of this problem you should assume special relativity holds. The results listed on the previous page may be helpful.

The neutrinos did not all arrive at once. The first arrived with an energy of about $E_1 = 40\,\text{MeV}$, and the last arrived about $t = 10\,\text{s}$ later with an energy of about $E_2 = 20\,\text{MeV}$.

b. One explanation of these observations is that neutrinos have a small mass m, so that when they have energy $E \gg mc^2$, their speed v is slightly slower than the speed of light.

   i. Find an approximate expression for $c - v$, to leading nontrivial order in mc^2/E.

   ii. Using the information above, numerically estimate the neutrino mass m, in units of eV/c^2.

c. Another explanation is that the neutrinos did not travel in straight lines, but rather were deflected by the intergalactic magnetic field. Suppose this field is uniform, $B = 10^{-13}\,\text{T}$, and directed perpendicular to the line joining Earth and the supernova, and that neutrinos have charge $q = \epsilon e$.

   i. If a neutrino has momentum p, then in the presence of the magnetic field, it travels in a circle of radius $r = p/(qB) \gg d$, and its path to the Earth has a total length ℓ. Find an approximate expression for $\ell - d$, to leading nontrivial order in d/r.

   ii. Using the information above, and assuming the neutrino mass is very small so that the effect in part b is negligible, numerically estimate ϵ.