AAPT UNITED STATES PHYSICS TEAM
AIP 2024

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination has two parts. Each part has three questions and lasts for 90 minutes.
- For each student, print out one copy of the exam and one copy of the answer sheets. Print everything single-sided, and do not staple anything. Divide the exam into the instructions (pages 2–3), Part A questions (pages 4–7), and Part B questions (pages 8–10). Page 11 is a graph page, in case students wish to use it as an additional page for problem B3.
- Begin by giving students the instructions and all of the answer sheets. Let the students read the instructions and fill out their information on the answer sheets. They can keep the instructions for both parts of the exam. Also give students blank sheets of paper to use as scratch paper throughout the exam.
- Students may bring calculators, but they may not use symbolic math, programming, or graphing features of these calculators. Calculators may not be shared, and their memory must be cleared of data and programs. Cell phones or other electronics may not be used during the exam or while the exam papers are present. Students may not use books or other references.
- To start the exam, collect signed Honesty Policy, give students the Part A questions, and allow 90 minutes to complete Part A. Do not give students Part B during this time, even if they finish with time remaining. At the end of the 90 minutes, collect the Part A questions and answer sheets.
- Then give students a 5 to 10 minute break. Then give them the Part B questions, and allow 90 minutes to complete Part B. Do not let students go back to Part A.
- At the end of the exam, collect everything, including the questions, the instructions, the answer sheets, and the scratch paper. Give them the Honor Code Certification and collect signed Codes. Students may *not* keep the exam questions.
- After the exam, sort each student's answer sheets by page number. Scan every answer sheet, including blank ones. Everybody may discuss the questions after April 3rd. Until April 19th, hold on to all of the answer sheets in the event that your scans are lost or illegible.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

Tengiz Bibilashvili, Kellan Colburn, Natalie LeBaron, Rishab Parthasarathy, Elena Yudovina, and Kevin Zhou.

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USA Physics Olympiad Exam

Instructions for the Student

- You should receive these instructions, the reference table on the next page, answer sheets, and blank paper for scratch work. Read this page carefully before the exam begins.
- You may use a calculator, but its memory must be cleared of data and programs, and you may not use symbolic math, programming, or graphing features. Calculators may not be shared. Cell phones or other electronics may not be used during the exam or while the exam papers are present. You may not use books or other outside references.
- When the exam begins, your proctor will give you the questions for Part A. You will have 90 minutes to complete three problems. Each question is worth 25 points, but they are not necessarily of the same difficulty. If you finish all of the questions, you may check your work, but you may not look at Part B during this time.
- After 90 minutes, your proctor will collect the questions and answer sheets for Part A. You may then take a short break.
- Then you will work on Part B. You have 90 minutes to complete three problems. Each question is worth 25 points. Do not look at Part A during this time. When the exam ends, you must return all papers to the proctor, including the exam questions.
- **Do not discuss the questions of this exam, or their solutions, until April 4. Violations of this rule may result in disqualification.**

Below are instructions for writing your solutions.

- All of your solutions must be written on the official answer sheets. Nothing outside these answer sheets will be graded. Before the exam begins, write your name, student AAPT number, and proctor AAPT number as directed on the answer sheets.
- There are several answer sheets per problem. If you run out of space for a problem, you may use the extra answer sheets, which are at the end of the answer sheet packet. To ensure this work is graded, you must indicate, at the bottom of your last answer sheet for that problem, that you are using these extra answer sheets.
- Only write within the frame of each answer sheet. To simplify grading, we recommend drawing a $\boxed{\text{box}}$ around your final answer for each subpart. You should organize your work linearly and briefly explain your reasoning, which will help you earn partial credit. You may use either pencil or pen, but sure to write clearly so your work will be legible after scanning.

Fundamental Constants

$g = 9.8$ N/kg

$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2

$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9$ N $\cdot$ m^2/C^2

$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}$ T $\cdot$ m/A

$c = 3.00 \times 10^8$ m/s

$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K

$N_\mathrm{A} = 6.02 \times 10^{23}$ (mol)$^{-1}$

$R = N_\mathrm{A} k_\mathrm{B} = 8.31$ J/(mol $\cdot$ K)

$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)

$e = 1.602 \times 10^{-19}$ C

1 eV $= 1.602 \times 10^{-19}$ J

$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s

$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2

Useful Approximations

$$(1+x)^n \approx 1 + nx + n(n-1)x^2/2 \text{ for } |nx| \ll 1$$
$$e^x \approx 1 + x + x^2/2 + x^3/6 \text{ for } |x| \ll 1$$
$$\sin\theta \approx \theta - \theta^3/6 \text{ for } |\theta| \ll 1$$
$$\cos\theta \approx 1 - \theta^2/2 \text{ for } |\theta| \ll 1$$

You may use this sheet for both parts of the exam.

End of Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Part A

Question A1

Ping Pong

A thin wire of negligible resistance and total length D is wound to form a thin cylindrical solenoid of length $\ell \ll D$. A conducting sphere of radius $R \ll \ell$ is attached to each end of the solenoid. Initially there is no current in the wire, and the spheres have charges Q and $-Q$. Give all your answers in terms of R, ℓ, D, and the speed of light $c = 1/\sqrt{\mu_0 \epsilon_0}$.

a. Assume this system can be modeled as an LC circuit. What is the angular frequency of its oscillations?

This system loses energy because it emits electromagnetic radiation. Consider an electric dipole consisting of charges $\pm q_0 \cos(\omega t)$ separated by distance d, whose dipole moment oscillates with amplitude $p_0 = q_0 d$. If d is much smaller than the wavelength λ of the radiation produced, then it can be shown that the power radiated is roughly (i.e. up to an order-one dimensionless factor)

$$P \sim \frac{\omega^4 p_0^2}{\epsilon_0 c^3}.$$

For the rest of the problem, your answers only need to be similarly rough estimates.

b. For this setup, the above formula applies if $D \gg D_0$. Find a rough estimate for D_0.

c. Assuming $D \gg D_0$, estimate the number of oscillations that occurs until half the energy is lost.

Question A2

Stellar Stability

A star in hydrostatic equilibrium has inward gravitational forces balanced by pressure gradients. Though the material in a star is not simply an ideal gas, in many cases its pressure P and density ρ are simply related by $P = K\rho^\gamma$ for constants K and γ. Throughout this problem, assume the star is spherically symmetric, its mass is conserved, and relativistic effects can be neglected.

a. A thin shell of the star at radius r_0 has density ρ_0 and thickness Δr, and experiences an inward gravitational field of magnitude g_0.

   i. What is the pressure difference ΔP_0 across the shell in equilibrium?

   ii. Suppose the entire star expands uniformly by a factor $1 + x$, so that the shell now has radius $r = r_0(1 + x)$. In terms of ΔP_0, x, and γ, what is the new pressure difference across it?

   iii. By considering the forces on the shell, write an expression for d^2r/dt^2 valid when x is small, in terms of g_0, γ, and x. For what values of γ will the star be stable?

Next, we consider some simple models of stars, where γ can be computed.

b. In a giant star, the pressure is $P = P_{\text{gas}} + P_{\text{rad}}$, where P_{gas} is due to the gas, which obeys the ideal gas law, and $P_{\text{rad}} \propto T^4$ is due to blackbody radiation. In the star's "radiation zone", P_{rad} is much larger than P_{gas}, but the two have a constant ratio. In this case, what is the value of γ?

c. A white dwarf is composed of electrons and nuclei. The electrons provide the outward pressure, while the nuclei cancel the electrons' charge, and are responsible for most of the mass density. Consider a region of a white dwarf where the number density of electrons is n_e.

   i. The electrons obey the Heisenberg uncertainty principle, $\Delta p\, \Delta x \gtrsim \hbar$, where Δx is the spacing between them, and Δp is the typical momentum that quantum mechanics implies they must have. Find a rough estimate for Δp in terms of n_e and $\hbar$.

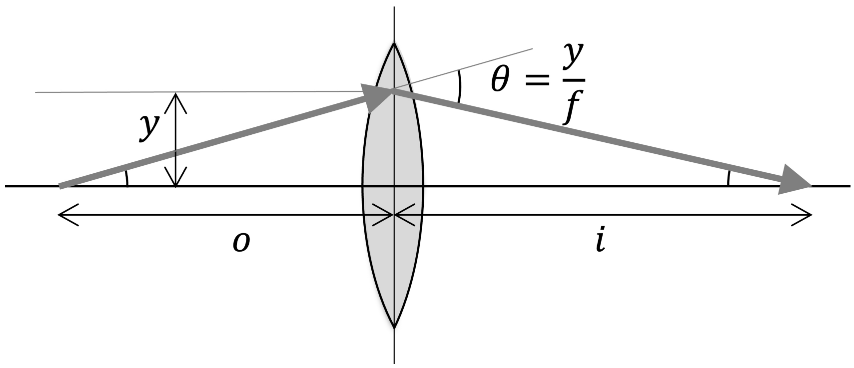
   ii. Using this result, find γ for a white dwarf.

   iii. A white dwarf has total mass M, radius R, and a relatively uniform density of order $\rho \sim M/R^3$. The radius is related to the mass by $R \propto M^n$ for a constant n. Find the value of n.

Question A3

Tilt Shift

An ideal converging lens of focal length f is centered at $x = y = 0$ with its axis of symmetry aligned with the x-axis. A light ray incident at height $y \ll f$ will be tilted inward by an angle $\theta = y/f$. In this problem, we will consider objects at $x = -o$, where $o > f$. The lens will produce a real image of the object at $x = i$, where $1/o + 1/i = 1/f$.



Even for an ideal lens, the image of a finite-sized object will generally be distorted.

a. Consider a pointlike object at $x = -o$ and $y = 0$. If it moves to the right a small distance δ_x, its image moves to the right a distance $m_x\,\delta_x$. If it moves up a small distance δ_y, its image moves up a distance $m_y\,\delta_y$. Find m_x and m_y in terms of i and o.

b. Suppose the object is a short stick, tilted an angle θ_o to the x-axis. In terms of i, o, and θ_o, what is the angle θ_i its image makes with the x-axis?

To produce a simple camera, we put the lens right next to a circular aperture of diameter $D \ll f$, and place a movable screen behind the lens. Suppose the location of the screen is chosen so that light from very distant objects will be focused to a point on the screen.

c. The light from a pointlike object at finite distance o will produce a finite-sized spot of radius r on the screen. Find r in terms of f, D, and o, assuming $o \gg f$.

d. If the camera primarily sees light of wavelength $\lambda \ll f, D$, find a rough estimate for the additional spread r_d of any image on the screen due to diffraction, in terms of f, D, and o.

e. Assuming the typical numbers $f = 5.0\,\text{cm}$, $D = 5.0\,\text{mm}$, and $\lambda = 500\,\text{nm}$, find the numeric values of o for which the blurring due to geometric effects exceeds the blurring due to diffraction.

Real photos are noisy because light is made of discrete photons, with energy $E = hc/\lambda$. Suppose the camera is illuminated uniformly with light of intensity $I = 1\,\text{W/m}^2$, its sensor has $N = 10^7$ pixels, and every photon passing through the aperture is detected, with equal probability, by one pixel in the sensor. This implies that if the expected number of photons arriving at a pixel on the sensor is n, the standard deviation of that number is $\sqrt{n}$.

f. If the aperture opens for time τ to take a photo, find the numeric value of τ for which the standard deviation of the brightness of each pixel is 1% of the mean.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you can review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

The Muon Shot

In 2023, American particle physicists recommended developing a muon collider to investigate the nature of fundamental particles. Such a collider requires much less space then other options because of the muon's high mass m, which makes it easier to accelerate to a very high energy $E \gg mc^2$.

a. When a muon colides head-on with an antimuon, which has the same energy and mass, a new particle of mass $2E/c^2$ can be produced. If the antimuon was instead at rest, what energy would the muon need to produce such a particle?

Unfortunately, muons and antimuons are unstable, with lifetime τ. That is, if one such particle exists at time $t = 0$, then in its rest frame, the probability it has not decayed by time t is $e^{-t/\tau}$.

b. Suppose the muons begin at rest, and are accelerated so that each muon's energy increases at a very large, constant rate α in the lab frame. Find the fraction f of muons that have decayed by the time each muon has energy E, assuming f is small.

The collider produces a "bunch" of muons with energy E, uniformly distributed in a thin disc of radius $R = 10^{-6}$ m. It also simultaneously produces a similar "antibunch" of antimuons. For simplicity, model each muon and antimuon as a sphere of radius $r = 10^{-21}$ m, and suppose a muon-antimuon collision occurs whenever two such spheres touch.

c. Initially, the bunch and antibunch each contain $N = 10^{14}$ particles. If they immediately collide head-on, what is the average number of muon-antimuon collisions, to one significant figure?

d. The bunch travels clockwise along a ring of circumference $\ell = 10$ km, while the antibunch travels along the same path in the opposite direction. Assume all particles maintain a constant energy $E = 10^5\, mc^2$, and that the muon lifetime is $\tau = 2.2 \times 10^{-6}$ s. To one significant figure, what is the average number of muon-antimuon collisions that occur before all of the particles decay?

Question B2

Solid Heat

In classical thermodynamics, a solid containing N atoms has a heat capacity $C_V = 3Nk_B$. The two parts of this question are independent. In both parts, we assume the solid has constant volume.

a. In a simple quantum model of a solid, the energy is $E = \hbar\omega m$, where m is the number of quanta and ω is a constant. Einstein showed that the entropy of such a solid is

$$\frac{S}{k_B} = (3N + m)\ln(3N + m) - m\ln(m)$$

up to a constant. According to the first law of thermodynamics, $dE = T\,dS$ for this system.

  i. Find an expression for m in terms of N and the quantity $\alpha = \hbar\omega/k_BT$.

  ii. We want to see how quantum effects modify the familiar classical result in the limit $\alpha \ll 1$, where the quantum corrections are small. Write an approximate expression for m, including terms of order α but neglecting terms of order α^2 or higher.

  iii. The heat capacity, with its leading quantum correction, is $C_V \approx 3Nk_B(1 + b\alpha^n)$ for some constants b and n. Find the values of b and n.

b. A vertical cylinder is filled with a monatomic ideal gas, and capped by a movable piston. The temperature is high enough for the piston to be modeled as a classical solid. The gas and piston contain the same number of atoms, but the mass of the gas is negligible compared to that of the piston. Assume the entire cylinder is in vacuum, and that the gas and piston do not transfer heat to their environment, but always remain in thermal equilibrium with each other.

  i. When the piston is in mechanical equilibrium, the column of gas has height h and pressure $P = P_0$. At this point, find dP/dh in terms of P_0 and h.

  ii. If the piston is given a small vertical impulse, what is the angular frequency of its subsequent oscillations? Give your answer solely in terms of h and the gravitational acceleration g.

Question B3

Quality Quest

The quality factor is a dimensionless number which quantifies how efficiently a system stores energy and how strongly it responds on resonance. For a circuit consisting of a capacitor C, an inductor L, and a small resistance R in series, the resonant frequency is approximately $\omega_0 = 1/\sqrt{LC}$, and the quality factor, assumed to be large throughout this problem, is

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

In this problem, we explore several ways to measure Q. Uncertainty analysis is not required.

a. Alice measures Q by seeing how oscillations in the circuit damp over time. Suppose that initially, the charge on the capacitor is q and the current is zero. The next time the current is zero, the charge is $-q(1-\delta)$. Find an approximate expression for δ, in terms of ω_0 and Q.

b. Bob and Charles drive their circuits with a sinusoidal voltage $V(t) = V_0 \cos \omega t$. It can be shown that in the steady state, the voltage across the capacitor oscillates with amplitude

$$V_c = \frac{V_0}{\sqrt{(1-\omega^2/\omega_0^2)^2 + (\omega/\omega_0 Q)^2}}.$$

The circuits Bob and Charles have are similar, but are not precisely the same.

i. Bob fixes the value of V_0 so that the highest value of V_c at any frequency is precisely 10.00 V. His equipment can precisely compare the amplitudes of a small DC and AC voltage. He thus performs two very accurate voltage measurements.

ω (rad/s)	0.0	183.3
V_c (Volts)	0.1219	0.1219

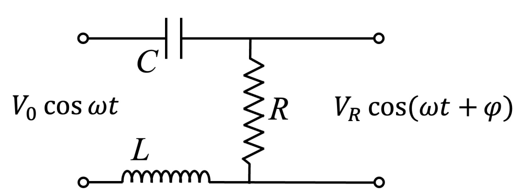
Using this data, find the numeric values of Q and ω_0 as accurately as possible.

ii. Charles can precisely tune ω, but cannot precisely measure small voltages. He thus fixes V_0 to some other value and takes data near the resonance, where V_c is relatively large.

ω (rad/s)	133.0	133.5	134.0	134.5	135.0	135.5	136.0	136.5	137.0
V_c (Volts)	3.64	4.76	6.52	8.53	8.18	6.06	4.44	3.42	2.75

Using this data, find the numeric values of Q and ω_0 as accurately as possible. (Hint: you may use the graph paper in the answer sheets, but full credit is attainable without graphing. To find Q, you should first find ω_0, then simplify the equation above using $\omega \approx \omega_0$.)

c. The gain function of this circuit is defined as $G = V_R/V_0$, where V_R is the amplitude of the voltage across the resistor, as shown below.



i. Find an expression for G in terms of ω, ω_0, and Q.

ii. This setup can be used to reject voltages at certain frequencies. Qualitatively describe the range(s) of frequencies for which G is small.

Page Problem Student Name: Student AAPT number:

Additional page: start additional page numbers from 14