T3. Physics of the Atmosphere (10 pts)

Part A. Surface temperature of the Earth (1.2 points)

A.1: The cross-section area receiving the solar radiation (falling as parallel rays) is πR_E^2, so taking into account the absorbed portion is a fraction $1-a$ of the total incident radiation, we find

$$P_0 = (1-a)\pi R_E^2 F_s.$$

Grading scheme for Task A.1.	Pts
Correct effective cross section area $A = \pi R_E^2$	0.1
Correct final answer	0.1
Total	**0.2**

Grading note: If the student uses a different cross sectional area, only 0.1 is given, provided it is the only mistake.

A.2: A black body radiates according to the Stefan-Boltzmann law, $P_{\text{bd}} = \sigma A T^4$, where σ is the Stefan-Boltzmann constant and A is the total surface area of the black body. At steady state

$$P_{\text{bd}} = P_0 \Rightarrow \sigma(4\pi R_E^2)T_{g0}^4 = (1-a)\pi R_E^2 F_s$$
$$\Rightarrow T_{g0} = \left((1-a)\frac{F_s}{4\sigma}\right)^{1/4} \approx 255\,\text{K} \approx -18\,^\circ\text{C}.$$

Grading scheme for Task A.2.	Pts
Energy balance	0.1
Correct explicit blackbody radiation formula, using the surface area of a sphere	0.1
Correct numerical value	0.1
Total	**0.3**

A.3: In the presence of the atmospheric layer, we write down the energy transfer balance in two regions: between the Earth's surface and the atmosphere, and between the atmosphere and outer space. Let the power radiated from Earth be P_E and the power radiated from each side of the atmosphere be P_{atmo}, then

$$P_E = P_{\text{atmo}} + t_{\text{sw}} P_0,$$
$$t_{\text{lw}} P_E + P_{\text{atmo}} = P_0.$$

Solving this system of equations and using $P_E = \sigma(4\pi R_E^2)T_g$ we find

$$T_g = \left(\frac{1+t_{\text{sw}}}{1+t_{\text{lw}}}\right)^{1/4} T_{g0} \approx 286\,\text{K} \approx 13\,^\circ\text{C}.$$

Grading scheme for Task A.3.	Pts
Statement on radiation balance in the region outside the atmosphere	0.1
Statement on radiation balance in the region between the atmosphere and Earth	0.2
Using t_{sw} correctly	0.1
Using t_{lw} correctly	0.1
Correct numerical result (if only analytical, then only 0.1)	0.2
Total	**0.7**

Part B. The absorption spectrum of atmospheric gases (1.8 points)

B.1: Let the natural (unstretched) length of the spring be l_0 and let x_A, x_B be the positions of particles A and B, respectively. The equation of motion of each particle due to the spring force can be written as:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_A = +\frac{k}{m_A}(\ell - \ell_0),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_B = -\frac{k}{m_B}(\ell - \ell_0),$$

where $\ell = x_B - x_A$ is the instantaneous length of the spring. Taking the difference of the two equations gives

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ell = -k\left(\frac{1}{m_B} + \frac{1}{m_B}\right)(\ell - \ell_0).$$

This is the equation of motion of a single effective particle attached to a spring with a spring constant k and an effective mass or reduced mass μ, given by:

$$\mu = \frac{1}{\frac{1}{m_A} + \frac{1}{m_B}} = \frac{m_A m_B}{m_A + m_B}.$$

Thus, the system undergoes a simple harmonic motion with an angular frequency:

$$\omega_d = \sqrt{\frac{k}{\mu}} = \sqrt{k\frac{m_A + m_B}{m_A m_B}}.$$

Grading scheme for Task B.1.	Pts
Writing down correct equations of motion for A and B (0.1 each)	0.2
Studying the equation of motion for x_A-x_B	0.1
Correct answer	0.2
Total	**0.5**

Grading note: A maximum of 0.2 points are given if the correct result is cited without justification.

B.2: The difference in energy between two consecutive levels in a quantum harmonic oscillator is given by $\hbar\omega$. So the energy of the photon is given by

$$E = \hbar\omega_d.$$

Grading scheme for Task B.2.	Pts
Correct result (Give 0.1 if h is used instead of $\hbar$. No other numerical factors receive credit.)	0.2
Total	**0.2**

B.3: The observed shift in the spectral line from f_0 is due to the Doppler effect. When the source is moving towards the observer with velocity v the frequency is shifted according to

$$f = f_0 \left(1 + v/c\right).$$

Thus, the shift in frequency is given by:

$$f - f_0 = \frac{v}{c} f_0.$$

Grading scheme for Task B.3.	Pts
Writing down an expression for Doppler effect (even if incorrect)	0.1
Correct answer	0.1
Total	**0.2**

B.4: To find the normalization constant C, we require that the total probability is equal to 1. This leads to:

$$\int_{-\infty}^{\infty} p(v)\,\mathrm{d}v = 1 \Rightarrow C \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{mv^2}{2k_BT}}\,\mathrm{d}v.$$

Using the integration formula provided, with $x = v$ and $a = \frac{m}{k_BT}$ we obtain:

$$C = \sqrt{\frac{m}{2\pi k_BT}}.$$

Grading scheme for Task B.4.	Pts
Normalization condition (even if done incorrectly from 0 to ∞)	0.1
Correct result	0.1
Total	**0.2**

B.5: Using the result of **B.4**, we obtain the following expression for the speed of a molecule in terms of the frequencies f and f_0:

$$v = \frac{f - f_0}{f_0} c.$$

We plug this back into the probability distribution formula to obtain:

$$p(f) \propto \exp\left[-\frac{mc^2}{2k_BT}\left(\frac{f - f_0}{f_0}\right)^2\right].$$

This gives the probability distribution for observing a molecule whose spectral line is Doppler shifted from f_0 to f.

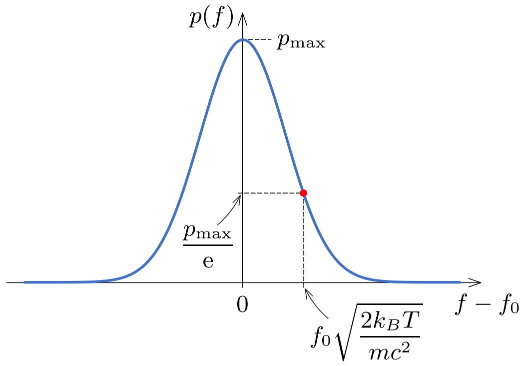
Grading scheme for Task B.5.	Pts
Replacing v by the Doppler effect result	0.1
Correct exponential dependence	0.2
Total	**0.3**

Grading note: If the student uses an incorrect Doppler effect formula, but one that matches their attempt in **B.3**, they get the 0.1 points.

B.6: The probability distribution $p(f)$ follows a Gaussian profile in the frequency shift $f - f_0$. The center of the profile is 0 and it drops to $1/e$ of its maximum value when the argument of the exponential is -1. This happens when

$$f^* - f_0 = f_0\sqrt{\frac{2k_BT}{mc^2}}.$$

The shape of the distribution can be seen in the figure below.



Grading scheme for Task B.6.	Pts
The distribution is has a single peak at zero	0.1
The distribution is symmetric	0.1
The distribution decays to zero on both ends	0.1
$f^\star - f_0$ is correct	0.1
Total	**0.4**

Part C. Stability of air in the atmosphere (2.7 points)

C.1: Consider a thin horizontal layer of thickness dz and surface area S. Since the air is in hydrostatic equilibrium, it's weight must be balanced by the difference in pressure forces. This results in the following relation:

$$p(z)S = p(z + \mathrm{d}z)S + \rho(z)gS\mathrm{d}z.$$

Simplifying and rearranging terms gives:

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho(z)g$$

The negative sign indicates a decrease in pressure with hight as expected.

Grading scheme for Task C.1.	Pts
Sum of forces equals zero	0.1
Correct pressure force above and below	0.1
Correct final answer	0.1
Total	**0.3**

C.2: Assuming we can treat air as an ideal gas, we can use the ideal gas law to express the density of air in terms of its pressure and temperature

$$pV = nRT \Rightarrow p(z)V = \frac{m}{\mu_{\text{air}}}RT(z).$$

Rewriting this in terms of the density gives:

$$\rho(z) = \frac{p(z)\mu_{\text{air}}}{RT(z)}.$$

Now we substitute the density expression into the expression obtained in **C.1**. This gives:

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\mu_{\text{air}} p(z)}{RT(z)} g$$

Grading scheme for Task C.2.	**Pts**
Ideal gas law	0.1
Correct final answer	0.1
Total	**0.2**

C.3: Assuming an isothermal atmosphere (i.e., constant temperature with altitude), $T(z) = T$, the equation simplifies to:

$$\frac{\mathrm{d}p}{p} = -\frac{\mu_{\text{air}}}{RT} g \mathrm{d}z.$$

Integrating both sides and assuming the pressure at height 0 is p_0 leads to:

$$\ln\left[\frac{p(z)}{p_0}\right] = -\frac{\mu_{\text{air}}}{RT} gz.$$

In a different form:

$$p(z) = p_0 \exp\left(-\frac{\mu_{\text{air}}}{RT} gz\right).$$

Grading scheme for Task C.3.	**Pts**
Recognizing a separable differential equation	0.1
Correct final answer	0.1
Total	**0.2**

C.4: Since the small mass of air is displaced adiabatically, it must satisfy the adiabatic condition for an ideal gas:

$$pV^{\gamma} = \text{const.},$$

where $\gamma = c_p/c_V$ is the adiabatic index, and c_p, c_V are the molar specific heats at constant pressure and volume respectively. Writing the volume in terms temperature and pressure using the ideal gas law gives:

$$p(T/p)^{\gamma} = \text{const.} \quad \Rightarrow \quad p^{1-\gamma} T^{\gamma} = \text{const.}$$

Taking the derivative of this expression with respect to the height z, we obtain:

$$(1-\gamma) p^{-\gamma} \frac{\mathrm{d}p}{\mathrm{d}z} T^{\gamma} + \gamma p^{1-\gamma} T^{\gamma-1} \frac{\mathrm{d}T}{\mathrm{d}z} = 0.$$

Simplifying and rearranging to have an expression for the adiabatic lapse rate gives:

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{1-\gamma}{\gamma} \frac{T(z)}{p(z)} \frac{\mathrm{d}p}{\mathrm{d}z}.$$

We now substitute the hydrostatic pressure gradient obtained in **C.3** to get:

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{1-\gamma}{\gamma} \frac{T(z)}{p(z)} \left[-\frac{p(z)\mu_{\text{air}}}{RT(z)} g\right] = \frac{1-\gamma}{\gamma} \frac{\mu_{\text{air}}}{R} g.$$

But $\gamma = c_p/c_V$, so

$$\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{1 - c_p/c_V}{c_p/c_V} \frac{\mu_{\text{air}}}{R} g = -\frac{\mu_{\text{air}}}{c_p} g,$$

where we used $c_p - c_V = R$.

This expression for the adiabatic lapse rate shows that the temperature drops linearly with height in an adiabatic atmosphere.

Grading scheme for Task C.4.	**Pts**
Writing the adiabatic relation in any form	0.1
Relating dT/dz to dp/dz	0.3
Correct final result	0.2
Total	**0.6**

C.4: To find the angular frequency of small oscillations of the air parcel, we begin by applying Newton's second law, where the primary forces acting on the parcel are buoyancy and gravity.

$$\delta m \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \rho_a(z) g \delta V - \delta m g,$$

where δm is the mass of the air parcel, δV is its volume and ρ_a is the density of the surrounding air. We can express the mass of the parcel in terms of its density ρ_p as $\delta m = \rho_p \delta V$. Substituting and simplifying gives:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{\rho_a(z+\delta z) - \rho_p(z+\delta z)}{\rho_p(z+\delta z)} g.$$

Assuming the parcel is at the same pressure as the atmosphere at $z + \delta z$, the density can be expressed in terms of temperature using the ideal gas law $\rho \propto 1/T$. This allows us to rewrite the last expression as:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{T_p(z+\delta z) - T_a(z+\delta z)}{T_a(z+\delta z)} g.$$

We can now express the temperature at $z+\delta z$ in terms of the lapse rates and the temperature at z using the definition $T(z+\delta z) = T(z) + \Gamma \delta z$. Therefore:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{T(z) + \Gamma \delta z - T(z) - \Gamma_a \delta z}{T(z) + \Gamma_a \delta z} g.$$

Simplifying the numerator and neglecting the infinitesimal term $\Gamma_a \delta z$ in the denominator gives:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{\Gamma - \Gamma_a}{T} g \delta z.$$

This is the equation of a simple harmonic oscillator, where the angular frequency is given by:

$$\omega = \sqrt{\frac{\Gamma_a - \Gamma}{T} g} = \sqrt{\frac{\mu_{\text{air}} g/c_p - \Gamma}{T} g}$$

The motion is stable whenever $\Gamma_a = \mu_{\text{air}} g/c_p > \Gamma$.

Grading scheme for Task C.5.	**Pts**
Inclusion of gravitational force with parcel density	0.2
inclusion of buoyancy force with air density	0.3
Correct equation of motion	0.2
Relating density to inverse temperature	0.2
Using appropriate approximation	0.2
Correct stability requirements	0.1
Correct angular frequency of small oscillation	0.2
Total	**1.4**

Part D. Moisture (2.7 points)

D.1: The change of entropy across a phase transition (evaporation in this case) is related to the latent heat of evaporation. If there was a mass m of liquid water, then $Q_{\text{evaporation}} = Lm$, then

$$\Delta S = \frac{Lm}{T}.$$

It is known that the volume of vapor is significantly larger than the volume of liquid of the same mass, therefore $\Delta V \approx V_{\text{vapor}}$, which can be found using the ideal gas law

$$V_{\text{vapor}} = \frac{nRT}{p_s(T)}.$$

The mass can be related to the number of moles n via $m = \mu_{\mathrm{H_2O}} n$, then

$$\frac{\mathrm{d}p_s}{\mathrm{d}T} = \frac{\mu_{\mathrm{H_2O}} L p_s}{RT^2}.$$

Grading scheme for Task D.1.	**Pts**
Correct entropy change	0.2
$V_{\text{vapor}} \gg V_{\text{liquid}}$	0.2
Correct final answer	0.1
Total	**0.5**

D.2: We can integrate the relationship found in **D.1** by separating variables to find

$$\ln\left[\frac{p_s(T)}{p_{so}}\right] = -\frac{\mu_{\mathrm{H_2O}} L}{R}\left(\frac{1}{T} - \frac{1}{T_o}\right).$$

Note that L is strictly a function of temperature, but we are assuming that L is a constant for the range of temperatures we investigate. Rearranging, we find

$$p_s(T) = p_{so}\exp\left[-\frac{\mu_{\mathrm{H_2O}} L}{R}\left(\frac{1}{T} - \frac{1}{T_o}\right)\right].$$

Grading scheme for Task D.2.	**Pts**
Recognizing a separable differential equation	0.1
Correct final answer	0.1
Total	**0.2**

D.3: Formation of liquid water happens when the partial pressure of water inside the parcel reaches the saturation pressure at a given temperature. The partial pressure of water vapor p_w can be related to the total pressure of the parcel p as

$$p_w = \frac{n_{\mathrm{H_2O}}}{n_{\text{air}}} p = \frac{m_{\mathrm{H_2O}}/\mu_{\mathrm{H_2O}}}{m_{\text{air}}/\mu_{\text{air}}} p = \phi \frac{\mu_{\text{air}}}{\mu_{\mathrm{H_2O}}} p.$$

Given that the air parcel is rising adiabatically, $p^{1-\gamma} T^{\gamma} = \text{const.}$, so

$$p(T) = p_i \left(\frac{T}{T_i}\right)^{c_p/R}.$$

Therefore, the transcendental equation that we need to solve is

$$\phi \frac{\mu_{\text{air}}}{\mu_{\mathrm{H_2O}}} p_i \left(\frac{T_l}{T_i}\right)^{\frac{c_p}{R}} = p_{so}\exp\left[-\frac{\mu_{\mathrm{H_2O}} L}{R}\left(\frac{1}{T_l} - \frac{1}{T_i}\right)\right].$$

This can be rearranged to get

$$T_l = \frac{1}{\dfrac{1}{T_i} - \dfrac{R}{\mu_{\mathrm{H_2O}} L}\ln\left[\phi \dfrac{\mu_{\text{air}}}{\mu_{\mathrm{H_2O}}}\dfrac{p_i}{p_{so}}\left(\dfrac{T_l}{T_i}\right)^{c_p/R}\right]}.$$

Substituting the numerical values, we get

$$T_l = \frac{1000\ \text{K}}{3.481 - 0.4695\ln\left(\dfrac{T_l}{290.15\ \text{K}}\right)}.$$

Solving this iteratively, we find $T \approx 286.8\,\text{K} \approx 13.7\,^{\circ}C$.

Grading scheme for Task D.3.	**Pts**
Using Dalton's law	0.4
correctly relating the moles ratio to mass ratio	0.2
Stating $p(T)$ for an adiabatic process	0.1
Understanding that partial pressure of water needs to reach saturation for condensation to start	0.5
Attempting to perform iterative search for the solution of the transcendental equation (by isolating T on one side)	0.4
Correct numerical solution	0.4
Total	**2.0**

Grading note: At most 0.4 pts can be given if the student does not use the partial pressure of water but uses the total pressure of air parcel.

Part E. Sun halo (1.6 points)

E.1: Using the notations of *Figure E*, the total angle of deviation δ can be written as the sum of the deviations in the two refractions:

$$\delta = \alpha - \alpha' + \beta - \beta'.$$

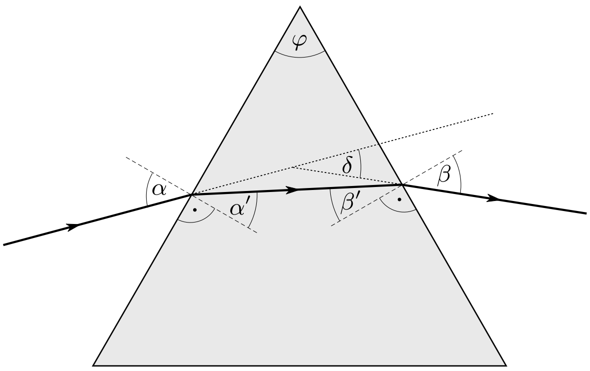


Figure E.

Consider the triangle of interior angles φ, $90° - \alpha'$ and $90° - \beta'$. Since the sum of these angles add up to 180°, we get

$$\varphi = \alpha' + \beta',$$

so δ simplifies to

$$\delta = \alpha + \beta - \varphi.$$

The relationship between α and α' (and similarly between β and β') is given by Snell's law:

$$\frac{\sin\alpha}{\sin\alpha'} = n, \qquad \frac{\sin\beta}{\sin\beta'} = n\,.$$

Expressing β in terms of α':

$$\sin\beta = n\sin\beta' = n\sin(\varphi - \alpha'),$$

From Snell's law α' can be written as

$$\alpha' = \arcsin\left(\frac{\sin\alpha}{n}\right).$$

Thus, β in terms of α is given by

$$\beta = \arcsin\left\{n\sin\left[\varphi - \arcsin\left(\frac{\sin\alpha}{n}\right)\right]\right\}.$$

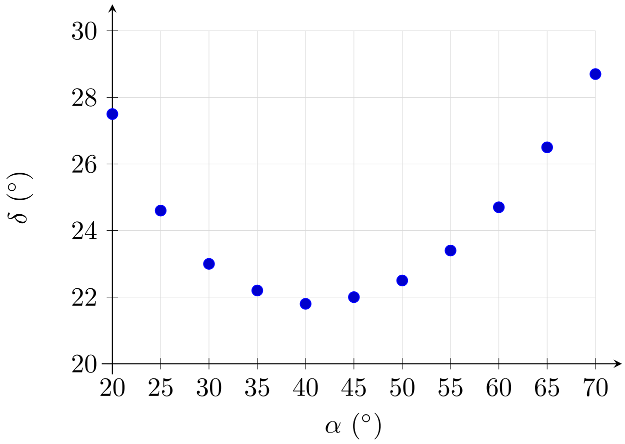
Finally, we get the result for δ:

$$\delta = \alpha + \arcsin\left\{n\sin\left[\varphi - \arcsin\left(\frac{\sin\alpha}{n}\right)\right]\right\} - \varphi.$$

Grading scheme for Task E.1.	**Pts**
Writing Snell's law for the two refractions (0.1 each)	0.2
Equation for δ in terms of α, β and φ	0.2
Using that $\alpha' + \beta' = \varphi$	0.1
Correct calculation leading to δ	0.2
Final formula for δ (any other equivalent form is acceptable)	0.1
Total	**0.8**

E.2: Notice that the situation corresponds to the case discussed in part **E.1** with $\varphi = 60°$. Here is the data table after substituting different values of α:

α	δ	α	δ
20°	27.5°	50°	22.5°
25°	24.6°	55°	23.4°
30°	23.0°	60°	24.7°
35°	22.2°	65°	26.5°
40°	21.8°	70°	28.7°
45°	22.0°	–	–



Grading scheme for Task E.2.	**Pts**
Substituting into the formula for δ correctly for all values of α (if at least 6 data points are calculated, 0.1 p can be given)	0.2
Data points are plotted in the correct graph	0.2
δ has a local minimum	0.2
Total	**0.6**

E.3: The minimum value of δ is around 21.8°, so that is the angle with respect to the direction of Sun where the halo appears.

Grading scheme for Task E.3.	**Pts**
Reading the minimal value of δ	0.1
Concluding that the angular size of halo corresponds to the minimal value of δ	0.1
Total	**0.2**