Notes on Grading Criterion: Full credit is awarded for a correct complex result that is clearly unattainable by simple estimation,

Part A

A.1)

The magnitude of the anticlockwise torque exerted by the left pan and the weights, resulting from an anticlockwise tilt of θ is given by

$(-b\sin\theta + l\cos\theta)m_1 g$ (1)

Criteria	Partial Credit
If Eq (1) is correctly obtained (The sign (+/−) does not affect the score.)	0.3

A.2)

The magnitude of the clockwise torque exerted by the left pan and the weights, resulting from an anticlockwise tilt of θ is given by

$(b\sin\theta + l\cos\theta)m_2 g$ (1)

criteria	partwise marks
If Eq (1) is correctly obtained (The sign (+/−) does not affect the score.)	0.3

A.3)

When the torques are in equilibrium, the beam balance remains stationary at a tilted position. Condition for rotational equilibrium:

$(-b\sin\theta_0 + l\cos\theta_0)m_1 g = (b\sin\theta_0 + l\cos\theta_0)m_2 g$ (1)

Rearranging for θ_0

$$\theta_0 = \tan^{-1}\left(\frac{(m_1 - m_2)l}{(m_1 + m_2)b}\right) \quad (2)$$

$$\tan\theta_0 = \frac{(m_1 - m_2)l}{(m_1 + m_2)b} \quad (2*)$$

criteria	partwise marks
If the torque equilibrium (Eq.(1))is correctly obtained	0.2
If the Eq.(2) or Eq.(2*) is correctly obtained	0.4

A.4)

The correct answer is ③. For a given set of weights, the balance beam tilts further as l increases and b decreases.

criteria	partwise marks
If option ③ is correctly selected,	0.3
If an incorrect answer is selected,	−0.1

A.5)

When equilibrium is reached at an inclination of θ, the magnitude of the clockwise torque about the pivot O is given by:

$m_2g(b\sin\theta + l\cos\theta)$

Meanwhile, the magnitude of the anticlockwise torque about the pivot O is as follows:

$m_1g(-b\sin\theta + l\cos\theta) + Mgd\sin\theta$ (1)

Condition for rotational equilibrium:

$(Md - m_1b)g\sin\theta + m_1gl_1\cos\theta = m_2gb\sin\theta + m_2gl\cos\theta$ (2)

By applying the equilibrium condition for the beam balance, the following relationship holds for the equilibrium angle θ_1.

$$\theta_1 = \tan^{-1}\left(\frac{(m_1 - m_2)l}{(m_1 + m_2)b - Md}\right) \quad (3)$$

$$\tan\theta_1 = \frac{(m_1 - m_2)l}{(m_1 + m_2)b - Md} \quad (3*)$$

criteria	partwise marks
If the expression for the anticlockwise torque (Eq.(1))is correctly obtained (The sign (+/−) does not affect the score.)	0.3
If the torque equilibrium condition(Eq.(2)) is correctly obtained	0.3
If Eq.(3) or Eq.(3*) is correctly obtained	0.8

A.6)

From the initial state, the beam begins to tilt toward the heavier side (m_1), and the following inequality holds:

$(Md - m_1 b) g \sin\theta + m_1 g l_1 \cos\theta > m_2 g b \sin\theta + m_2 g l \cos\theta$ (1)

If the coefficient of $\sin\theta$ satisfies $(Md - m_1 b) \ge m_2 b$, the left−hand side of the equation remains greater, causing the beam to undergo continuous angular acceleration without reaching equilibrium. Conversely, if $(Md - m_1 b) < m_2 b$ is met, there exists an angle at which the two torques achieve equilibrium.

The answer is:

$(Md - m_1 b) < m_2 b$ (2)

criteria	partwise marks
If the inequality including θ (Eq.(1)) is correctly obtained	0.2
If the final inequality for stable equilibrium (Eq.(2)) is correctly is correctly obtained	0.2

Part. B.

B.1)

The distance the heavier mass m_L descends and the lighter mass m_R ascends are both $l\sin\theta$ which is independent of x_R, x_L.

Due to the symmetry of the balance itself, its potential energy does not change. The potential energy of the entire system is given by

$(m_R - m_L) g l \sin\theta$ (1)

criteria	partwise marks
When the obtained potential energy differs from Eq. (1) by a constant	0,2
If Eq (1) is correctly obtained	0.3

B.2) Let θ be the rotation angle of the arms; the beam undergoes fixed-axis rotation, and the kinetic energy of the beam itself is:

$\frac{1}{2}(I_1 + I_2)\dot{\theta}^2$ (1)

The speed at the contact point between the pan and the arm is $l\dot{\theta}$, and the pan undergoes translational motion at the same speed without rotation. The weights on the pan also move at this speed, maintaining their orientation. Therefore, the kinetic energy of the pans and the weights is:

$$\frac{1}{2}(2m+m_L+m_R)l^2\dot{\theta} \qquad (2)$$

The total kinetic energy of the system, including the beam, is:

$$\frac{1}{2}(I_1+I_2)\dot{\theta}^2+\frac{1}{2}(2m+m_L+m_R)l^2\dot{\theta}^2 \qquad (3)$$

criteria	partwise marks
If the kinetic energy of the beam (Eq.(1)) is correctly obtained	0.2
If the kinetic energy of the pans and weights (Eq.(2)) is correctly obtained	0.3
If the kinetic energy of the total system (Eq.(3)) is correctly obtained	0.5

B.3)

The total mechanical energy of the system is:

$$\frac{1}{2}(I_1+I_2)\dot{\theta}^2+\frac{1}{2}(2m+m_L+m_R)l^2\dot{\theta}^2+(m_R-m_L)gl\sin\theta=C \qquad (1)$$

The equation of motion obtained by differentiating with respect to time is:

$$\left(I_1+I_2+(2m+m_L+m_R)l^2\right)\ddot{\theta}=(m_L-m_R)gl\cos\theta \qquad (2)$$

criteria	partwise marks
Correct expression for the conservation of mechanical energy (Eq.(1)) is correctly obtained	0.2
Attempting to differentiate Eq. (1) with respect to time	0.2
If the equation of motion (Eq.(2)) is correctly obtained	0.6

B.4)

The relationship between the forces acting on the left pan and its acceleration is:

$$(m+m_L)g-(T_{L1}+T_{L2})=(m+m_L)l\ddot{\theta} \qquad (1)$$

Eliminating $\ddot{\theta}$ using the relationship given in the problem for the horizontal position results in:

$$T_{L1}+T_{L2}=(m+m_L)g-(m+m_L)l\ddot{\theta}=(m+m_L)g-\frac{(m+m_L)l^2(m_L-m_R)g}{\left(I_1+I_2+(2m+m_L+m_R)l^2\right)}$$

(2)

$$=(m+m_L)g\frac{(I_1+I_2+(2m+2m_R)l^2)}{\left(I_1+I_2+(2m+m_L+m_R)l^2\right)} \qquad (2*)$$

Meanwhile, the torque relationship for the upper beam is:

$$(T_{L1}-T_{R1})l=I_1\ddot{\theta} \qquad (3)$$

Eliminating $\ddot{\theta}$ using the given relationship for the horizontal position results in:

$$T_{L1}-T_{R1}=\frac{I_1(m_L-m_R)g}{\left(I_1+I_2+(2m+m_L+m_R)l^2\right)} \qquad (4)$$

$$T_{L2}+T_{R1} \qquad (5)$$

$$=(m+m_L)g\frac{(I_1+I_2+(2m+2m_R)l^2)}{\left(I_1+I_2+(2m+m_L+m_R)l^2\right)}-\frac{I_1(m_L-m_R)g}{\left(I_1+I_2+(2m+m_L+m_R)l^2\right)}$$

criteria	partwise marks
If Eq (1) is correctly obtained	0.2
If Eq (2) or Eq.(2*) is correctly obtained	0.2
If Eq (3) is correctly obtained	0.2
If Eq (4) is correctly obtained	0.2
If Eq (5) is correctly obtained	1

B.5)

No values can be determined.

From the translational motion of the two pans and the rotational motion of the beam, the forms of four linear simultaneous equations can be obtained with T_{L1}, T_{L2}, T_{R1}, T_{R2} as unknowns.

$$T_{L1}-T_{R1}=A \qquad (1)$$

$$T_{L2}-T_{R2}=B \qquad (2)$$

$$T_{L1}+T_{L2}=C \qquad (3)$$

$$T_{R1}+T_{R2}=D \qquad (4)$$

Here, $A, B, C,$ and D are values expressed in terms of the variables and parameters given in the problem. Since their specific calculated values are not crucial to the form of the equation, they are represented simply as constants $A, B, C,$ and D. Since the four equations are not independent (no unique solution), T_{L1}, T_{L2}, T_{R1}, T_{R2} cannot be determined individually.

Finding the force exerted by the pivot on the upper beam is equivalent to finding $T_{L1}+T_{R1}$. If this were found, then T_{L1} and T_{R1} could be determined. Since $T_{L1}$and T_{R1} cannot be found, $T_{L1}+T_{R1}$ also cannot be determined.

criteria	partwise marks
For correctly determining each value	0.1(Max 0.2 pt)
penalty for each incorrect determination	-0.1
For each of the four equation forms (Eq.(1)~(4)) correctly obtained	0.1 (maximum of 0.4 points, 0.1×4)

B. 6)

Let the acceleration of the entire system (balance and weights) be a_{cm}:

$$N-(M_T+m_R+m_L)g=(M_T+m_R+m_L)a_{cm}$$

$$N-(M_T+m_R+m_L)g=(m+m_R)l\ddot{\theta}-(m+m_L)l\ddot{\theta} \qquad (1)$$

Therefore, the normal force N is:

$$N=(M_T+m_R+m_L)g+(m_R-m_L)l\ddot{\theta} \qquad (1*)$$

$$=(M_T+m_R+m_L)g-\frac{(m_L-m_R)^2 g}{\left(I_1+I_2+(2m+m_L+m_R)l^2\right)} \qquad (2)$$

criteria	partwise marks
If Eq. (1) or Eq (1*) is correctly obtained	0.4
If Eq (2) is correctly obtained	0.6

Part. C.

C.1)

The conservation of energy when tilted by an angle θ anticlockwise is:

$$\frac{1}{2}(I_1+I_2)\dot{\theta}^2+\frac{1}{2}(m_1+m_2)l^2\dot{\theta}^2+(m_2-m_1)gl\sin\theta+Mdg(1-\cos\theta)=Const.$$

(1)

The equation of motion obtained by differentiating with respect to time is:

$$\left(I_1+I_2+(m_1+m_2)l^2\right)\ddot{\theta}=(m_1-m_2)gl\cos\theta-Mgd\sin\theta \quad (2)$$

Thus,

$$A=\left(I_1+I_2+(m_1+m_2)l^2\right), B=(m_1-m_2)gl, C=-Mgd$$

criteria	partwise marks
Conservation of mechanical energy (Eq.(1)) is correctly obtained	0.6 (potential energy part in Eq.(1) is 0.2 pt) (kinetic energy part in Eq.(1) is 0.2 pt)
Attempting to differentiate Eq. (1) with respect to time	0.2
If A is correctly obtained (Full credit is given if the values of A, B, and C are scaled by the same factor.)	0.4
If B is correctly obtained (Full credit is given if the values of A, B, and C are scaled by the same factor.)	0.4
If C is correctly obtained (Full credit is given if the values of A, B, and C are scaled by the same factor.)	0.6

C.2)

The equation of motion is:

$$\left(I_1+I_2+(m_1+m_2)l^2\right)\ddot{\theta}=(m_1-m_2)gl\cos\theta-Mgd\sin\theta$$

At equilibrium, $\ddot{\theta}=0$.

Thus, $(m_1 - m_2)gl\cos\theta_0 - Mgd\sin\theta_0 = 0$ (1)

$\tan\theta_0 = \dfrac{(m_1-m_2)l}{Md}$ (1*),

$\sin\theta_0 = \dfrac{(m_1-m_2)l}{\sqrt{(m_1-m_2)^2 l^2 + M^2 d^2}}$ (2)

$\cos\theta_0 = \dfrac{Md}{\sqrt{(m_1-m_2)^2 l^2 + M^2 d^2}}$ (3)

The form of the equation:

$A\ddot{\theta} = B\cos\theta + C\sin\theta$,

$A = \left(I_1 + I_2 + (m_1+m_2)l^2\right), B = (m_1 - m_2)gl, C = -Mgd$

Expressing the rotation angle as $\theta = \theta_0 + \eta$:

$A\ddot{\eta} = B\cos(\theta_0 + \eta) + C\sin(\theta_0 + \eta)$

$= B(\cos\theta_0\cos\eta - \sin\theta_0\sin\eta) + C(\sin\theta_0\cos\eta + \cos\theta_0\sin\eta)$ (4)

$\simeq (B(\cos\theta_0 - \eta\sin\theta_0) + C(\sin\theta_0 + \eta\cos\theta_0)$ (5)

$= (C\cos\theta_0 - B\sin\theta_0)\eta + B\cos\theta_0 + C\sin\theta_0$ (5*)

Combining Eq. (2) and Eq. (3), we obtain,

$C\cos\theta_0 - B\sin\theta_0 = -g\sqrt{(m_1-m_2)^2 l^2 + M^2 d^2}$ (6)

$B\cos\theta_0 + C\sin\theta_0 = 0$ (7)

$\left(I_1 + I_2 + (m_1+m_2)l^2\right)\ddot{\eta} = -g\sqrt{(m_1-m_2)^2 l^2 + M^2 d^2}\,\eta$ (8)

Even if a student fails to determine A, B, and C in Question C.1, the following conclusion can be obtained if they carry out the calculations correctly while leaving A, B, and C as variables/parameters.

$A\ddot{\eta} = -\sqrt{(B^2 + C^2}\,\eta$ (8*)

criteria	partwise marks
If Eq.(8) or (8*) correctly obtained	1.9
If Eq.(1) or Eq.(1*) is correctly obtained	0.2
If $\sin\theta_0$ (Eq.(2)) is correctly obtained	0.2
If $\cos\theta_0$ (Eq.(3)) is correctly obtained	0.2
If Eq.(5) or Eq.(5*) is correctly obtained using approximation	0.5 (0.3 points for correctly obtaining Eq. (4) without approximation)
If Eq.(6) is correctly obtained	0.3

If Eq.(7) is correctly obtained	0.3

C.3)

Since the restoring torque is approximated as simple harmonic motion proportional to η, the period of simple harmonic oscillation is as follows:

$$T = 2\pi\sqrt{\frac{(I+(m_1+m_2)l^2)}{g\sqrt{(m_1-m_2)^2l^2+M^2d^2}}} \qquad (1)$$

The period is maximized under the condition $m_1 = m_2$

For the case $m_L = m_R = 0$ (2)

$$T = 2\pi\sqrt{\frac{I}{Mdg}} \qquad (3)$$

criteria	partwise marks
If Eq.(1) is correctly obtained	0.2
Identifying the condition $m_1 = m_2$	0.2
If Eq.(3) is correctly obtained	0.2