2026 Theory Problem Q3

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## Solution to A.1

The equation is easily solved if we define $\tilde{q} = q - f_0/\omega_0^2$ for $0 < t < T_0/2$. The equation becomes

$$\ddot{\tilde{q}} + \omega_0^2 \tilde{q} = 0$$

For $T_0/2 < t < T_0$, we may define $\bar{q} = q + f_0/\omega_0^2$ to obtain

$$\ddot{\bar{q}} + \omega_0^2 \bar{q} = 0$$

Before turning on the external force $f(t)$, the motion was given as

$$q_{t<0}(t) = A \sin(\omega_0 t + \delta)$$

$A$ is related to the initial energy, $E = \frac{1}{2} m \omega_0^2 A^2$ and the other initial condition (i.e. the value $q(0)$) is given by $\delta$. For $0 < t < T_0/2$, the general solution is

$$q_{0<t<T_0/2}(t) = \tilde{A} \sin(\omega_0 t + \tilde{\delta}) + f_0/\omega_0^2$$

Equating the configuration at $t = 0$, we have

$$\begin{aligned} A \sin \delta &= \tilde{A} \sin \tilde{\delta} + f_0/\omega_0^2 \\ A \cos \delta &= \tilde{A} \cos \tilde{\delta} \end{aligned}$$

Now we turn to the motion during $T_0/2 < t < T_0$. Then the general solution is

$$q_{T_0/2<t<T_0} = \bar{A} \sin(\omega_0 t + \bar{\delta}) - f_0/\omega_0^2$$

And equating the configuration at $t = T_0/2$, using $\omega_0 T_0 = 2\pi$ we have

$$-\tilde{A}\sin\tilde{\delta} + f_0/\omega_0^2 = -\bar{A}\sin\bar{\delta} - f_0/\omega_0^2$$
$$\tilde{A}\cos\tilde{\delta} = \bar{A}\cos\bar{\delta}$$

Now at $t = T_0$, the position and velocity is

$$\begin{aligned}
q(T_0) &= \bar{A}\sin\bar{\delta} - f_0/\omega_0^2 \\
&= \tilde{A}\sin\tilde{\delta} - 3f_0/\omega_0^2 \\
&= A\sin\delta - 4f_0/\omega_0^2 \qquad (1) \\
\dot{q}(T_0) &= \bar{A}\omega_0\cos\bar{\delta} \\
&= \tilde{A}\omega_0\cos\tilde{\delta} \\
&= A\omega_0\cos\delta \qquad (2)
\end{aligned}$$

| Question part | Total marks | Partial marks | Explanation for partial marks and special cases |
|---|---|---|---|
| A.1 | 1.2 | 0.6 | $q(T_0) = A\sin\delta - 4f_0/\omega_0^2$ correct |
| | | 0.6 | $\dot{q}(T_0) = A\omega_0\cos\delta$ correct |

## Solution to A.2

The total mechanical energy is

$$\begin{aligned}
m^{-1}E_{t\geq T_0} &= \frac{1}{2}\bar{A}^2\omega_0^2\cos^2\bar{\delta} + \frac{1}{2}\omega_0^2(\bar{A}\sin\bar{\delta} - f_0/\omega_0^2)^2 \\
&= \frac{1}{2}\tilde{A}^2\omega_0^2\cos^2\tilde{\delta} + \frac{1}{2}\omega_0^2(\bar{A}\sin\bar{\delta} + f_0/\omega_0^2)^2 - 2\bar{A}f_0\sin\bar{\delta} \\
&= \frac{1}{2}A^2\omega_0^2\cos^2\delta + \frac{1}{2}\omega_0^2(\tilde{A}\sin\tilde{\delta} - f_0/\omega_0^2)^2 - 2\bar{A}f_0\sin\bar{\delta} \\
&= \frac{1}{2}A^2\omega_0^2\cos^2\delta + \frac{1}{2}\omega_0^2(\tilde{A}\sin\tilde{\delta} + f_0/\omega_0^2)^2 - 2\tilde{A}f_0\sin\tilde{\delta} - 2\bar{A}f_0\sin\bar{\delta} \\
&= m^{-1}E_{t\leq 0} - 2f_0(2\tilde{A}\sin\tilde{\delta} - 2f_0/\omega_0^2) \\
&= m^{-1}E_{t\leq 0} - 4f_0(A\sin\delta - 2f_0/\omega_0^2)
\end{aligned}$$

We thus have $\Delta E = -4mf_0(A\sin\delta - 2f_0/\omega_0^2) = 4mf_0(2f_0/\omega_0^2 - A\sin\delta)$.

| Question part | Total marks | Partial marks | Explanation for partial marks and special cases |
|---|---|---|---|
| A.2 | 1.2 | 1.2 | $\Delta E = 4mf_0(2f_0/\omega_0^2 - A\sin\delta)$ correct |

## Solution to A.3

Since $\langle \sin\delta\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\theta d\theta = 0$ and $\langle \sin^2\delta\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2\theta d\theta = \frac{1}{2}$,

$$\langle \Delta E\rangle = \frac{8mf_0^2}{\omega_0^2}$$
$$\langle (\Delta E)^2\rangle = 8m^2f_0^2A^2 + 64m^2f_0^4/\omega_0^4$$

| Question part | Total marks | Partial marks | Explanation for partial marks and special cases |
|---|---|---|---|
| A.3 | 1.2 | 0.6 | $\langle \Delta E\rangle = 8f_0^2m/\omega_0^2$ correct |
| | | 0.6 | $\langle (\Delta E)^2\rangle = 8f_0^2m^2(A^2 + 8f_0^2/\omega_0^4)$ correct |

## Solution to B.1

Since $\dot{S}_\pm = \pm i\omega_2 e^{\pm i\omega_2 t}\Sigma_\pm + e^{\pm i\omega_2 t}\dot{\Sigma}_\pm$ and $\dot{S}_z = \dot{\Sigma}_z$, the equations for $\Sigma_\pm, \Sigma_z$ are

$$\dot{\Sigma}_+ = -i(\omega_0 + \omega_2)\Sigma_+ + i\omega_1\Sigma_z$$
$$\dot{\Sigma}_- = +i(\omega_0 + \omega_2)\Sigma_- - i\omega_1\Sigma_z$$
$$\dot{\Sigma}_z = \frac{i\omega_1}{2}(\Sigma_+ - \Sigma_-)$$

Then there is no explicit time-dependent function any more. Using the definition $\Sigma_\pm = \Sigma_x \pm i\Sigma_y$, the above equations become

$$\begin{aligned}\dot{\Sigma}_x &= (\omega_0 + \omega_2)\Sigma_y \\ \dot{\Sigma}_y &= -(\omega_0 + \omega_2)\Sigma_x + \omega_1\Sigma_z \\ \dot{\Sigma}_z &= -\omega_1\Sigma_y\end{aligned}$$

In other words, $M_x = -\omega_1, M_y = 0, M_z = -\omega_0 - \omega_2$.

| Question part | Total marks | Partial marks | Explanation for partial marks and special cases |
|---|---|---|---|
| B.1 | 1.5 | 0.5 | $M_x = -\omega_1$ correct |
| | | 0.5 | $M_y = 0$ correct |
| | | 0.5 | $M_z = -\omega_0 - \omega_2$ correct |

## Solution to B.2

As we perform rotation of the axes, the vector in the original frame has components $(v_x, v_y, v_z)$ and after the rotation the same vector has coordinates $(v_X, v_Y, v_Z)$. When the general relation is $v_x = v_X \cos\Theta + v_Z \sin\Theta, v_y = v_Y, v_z = -v_X \sin\Theta + v_Z \cos\Theta$ and as we apply it to $\vec{\Sigma}$, we obtain

$$\dot{\Sigma}_X = M_y\Sigma_Z - [M_z \cos\Theta + M_x \sin\Theta]\,\Sigma_Y \tag{3}$$

$$\dot{\Sigma}_Y = [M_z \cos\Theta + M_x \sin\Theta]\,\Sigma_X + [M_z \sin\Theta - M_x \cos\Theta]\,\Sigma_Z \tag{4}$$

$$\dot{\Sigma}_Z = -M_y\Sigma_X + [-M_z \sin\Theta + M_x \cos\Theta]\,\Sigma_Y \tag{5}$$

If we further substitute the answer to B.1 and express the equations using $\omega_0, \omega_1, \omega_2$ instead of $M_x, M_y, M_z$,

$$\dot{\Sigma}_X = [(\omega_0 + \omega_2)\cos\Theta + \omega_1 \sin\Theta]\,\Sigma_Y \tag{6}$$

$$\dot{\Sigma}_Y = -\,[(\omega_0 + \omega_2)\cos\Theta + \omega_1 \sin\Theta]\,\Sigma_X + [-(\omega_0 + \omega_2)\sin\Theta + \omega_1 \cos\Theta]\,\Sigma_Z \tag{7}$$

$$\dot{\Sigma}_Z = [(\omega_0 + \omega_2)\sin\Theta - \omega_1 \cos\Theta]\,\Sigma_Y \tag{8}$$

| Question part | Total marks | Partial marks | Explanation for partial marks and special cases |
|---|---|---|---|
| B.2 | 0.9 | 0.3 | The equation for $\dot{\Sigma}_X$ Eq.(3) correct 0.2pt if the answer to B.1 is already substituted and gives Eq.(6) |
| | | 0.3 | The equation for $\dot{\Sigma}_Y$ Eq.(4) correct 0.2pt if answer to B.1 is already substituted and gives Eq.(7) |
| | | 0.3 | The equation for $\dot{\Sigma}_Z$ Eq.(5) correct 0.2pt if the answer to B.1 is already substituted and gives Eq.(8) |

## Solution to B.3

Now we choose $\tan\Theta = \omega_1/(\omega_0+\omega_2)$. Then the above equation is simplified to give

$$\begin{aligned}\dot{\Sigma}_X &= +\Omega\Sigma_Y\\ \dot{\Sigma}_Y &= -\Omega\Sigma_X\\ \dot{\Sigma}_Z &= 0\end{aligned}$$

where $\Omega = \sqrt{(\omega_0+\omega_2)^2+\omega_1^2}$.

| Question part | Total marks | Partial marks | Explanation for partial marks and special cases |
|---|---|---|---|
| B.3 | 1.0 | 0.5 | $\tan\Theta = \omega_1/(\omega_0+\omega_2)$ correct |
| | | 0.5 | $\Omega = \sqrt{(\omega_0+\omega_2)^2+\omega_1^2}$ correct |

## Solution to B.4 and B.5

Since the equations for different spins (with distinct initial conditions) are all independent of each other, we can simply solve the equations for a single spin with initial condition $S_x(0) = S_y(0) = 0$ and $S_z(0) = \mu \neq 0$.

1. **Relationship for $S_z$**

By definition, $S_z$ is equal to $\Sigma_z$, which can be expressed as:

$$S_z(t) = -\Sigma_X(t)\sin\Theta + \Sigma_Z(t)\cos\Theta \tag{9}$$

**2. Substituting $\Sigma_X(t)$ and $\Sigma_Z(t)$**

Applying the initial conditions $S_z(0) = \mu$ and $S_x(0) = S_y(0) = 0$, the solutions to the differential equations are:

$$\dot{\Sigma}_Z = 0 \implies \Sigma_Z(t) = \mu\cos\Theta \tag{10}$$

$$\ddot{\Sigma}_X = -\Omega^2\Sigma_X \implies \Sigma_X(t) = -\mu\sin\Theta\cos(\Omega t) \tag{11}$$

Substituting these two solutions into the equation from step 1 and by recalling the linearity of the equation:

$$\langle S_z(t)\rangle = \langle S_z(0)\rangle\left(\sin^2\Theta\cos(\Omega t) + \cos^2\Theta\right) \tag{12}$$

The given condition is satisfied only if $\Theta = \pi/4$ and $\Omega T_1 = \pi$. Since $\tan\Theta = \omega_1/(\omega_0+\omega_2) = 1$, $\Omega = \sqrt{\omega_1^2 + (\omega_0+\omega_2)^2} = \sqrt{2}\omega_1$. We thus have $\omega_1 T_1 = \frac{\sqrt{2}\pi}{2}$.

| Question part | Total marks | Partial marks | Explanation for partial marks and special cases |
|---|---|---|---|
| B.4 | 1.5 | 1.5 | The solution for $\langle S_z(t)\rangle$ Eq.(12) correct Or mathematically equivalent form via the answer of B.3, e.g. $\frac{\langle S_z(0)\rangle}{\omega_1^2+(\omega_0+\omega_2)^2}\left[\omega_1^2\cos(t\sqrt{(\omega_0+\omega_2)^2+\omega_1^2}) + (\omega_0+\omega_2)^2\right]$ |
| B.5 | 1.5 | 1.5 | $\omega_1 T_1 = \pi/\sqrt{2} = \sqrt{2}\pi/2$ correct |