QUESTION T2 [10 pts]. Cooking with sunlight?

The redistribution of luminous flux into regions of high and low intensity due to reflection or refraction can lead to the formation of *caustics:* curves formed by the intersection of infinitesimally close light rays. As an example, Figure 2a shows the caustic formed by rays incident paraxially on a half-cylindrical mirror. Figure 2b shows a photograph of the caustic formed by the reflection of light on the bottom part of a coffee cup. The brightest and central point of the caustic is called the *cusp*; at this point, the light intensity is maximal.

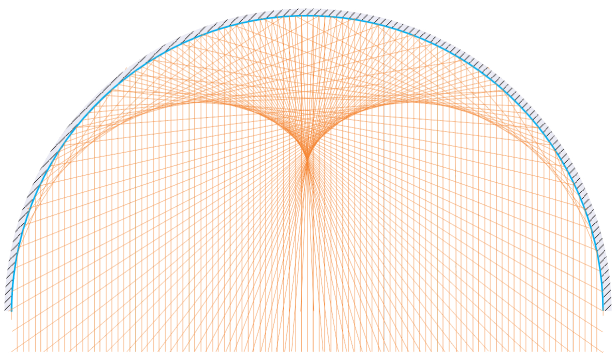

Fig. 2a.

Fig. 2b.

Caustics have a variety of applications in science, ranging from high-precision optics and quantum optics to astrophysics. In this question, you will study the caustic formation by light reflection on a half-cylinder mirror of radius R, as shown in figure 2c. You will also investigate other aspects of this mirror, such as its application in *solar cookers*.

Throughout the problem, assume that the incident rays come from a distant external light source and are parallel to the mirror's optical axis. By symmetry, it is sufficient to study reflection in one plane. Figure 2d shows such a plane, the mirror, and the coordinate system used throughout the problem.

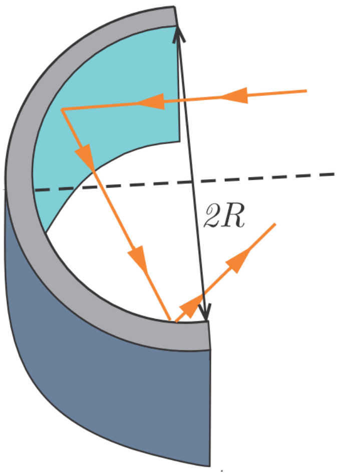

Fig. 2c.

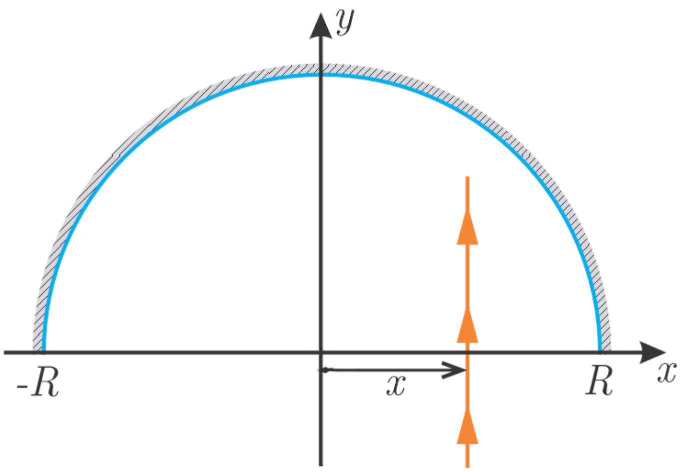

Fig. 2d.

T2-A [1.5 pts]. Multiple reflections

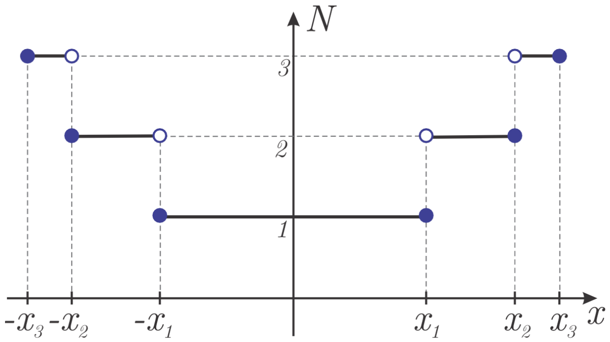

Fig.2e.

Figure 2e shows the relationship between the x-coordinate of an incident ray and the number N of reflections it undergoes, up to $N = 3$. As x varies over the interval $(-R, R)$, the variable N varies over all natural numbers greater than or equal to 1. The maximum distance from the optical axis that allows an incident ray to be reflected by the mirror at most N times is denoted x_N.

T2-A1 [1.5 pts]. Find the general expression for x_N in the infinite sequence $x_1, x_2, x_3, \ldots$ that limits the coordinates of the rays with at most N reflections. Express your answer in terms of R and N.

T2-B [4.0 pts]. Solar Cooker

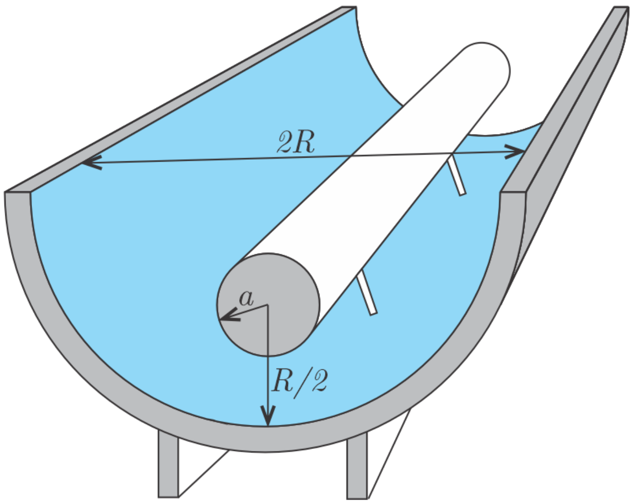

Fig.2f.

Consider a system consisting of a half-hollow-cylinder mirror and a cylindrical container. The cylinder is fully absorbing: any ray of light hitting it will be absorbed. The mirror radius is R. The container radius a. The axes of the mirror and the container are parallel. The center of the container is $R/2$ from the mirror on the system's symmetry plane, as shown in figure 2f. The mirror harnesses solar radiation to heat food inside the container. Assume that the sunlight is of constant and uniform intensity (power per unit area), and that the light rays are parallel to the optical axis of the mirror. Furthermore, assume that a is such that any ray that is absorbed by the cylinder reflects from the mirror at most once. Let $\theta_{\max}$ be the maximum angle of incidence on the mirror (measured with respect to the normal drawn at the point of incidence) of any reflected ray striking the container.

T2-B1 [2.0 pts]. The container radius is $a = \alpha \sin(\theta_{max}) + \beta \sin(2\theta_{max})$. Write α and β in terms of R.

If you do not find α and β, use the expression given for the rest of the problem.

T2-B2 [1.5 pts]. Let P_0 be the power that would be received by the cyllinder if the mirror was not present. Write P/P_0, the ratio of the actual received power P to P_0, in terms of $\theta_{\max}$.

T2-B3 [0.5 pts]. If $R = 1.0$ m, write the value of a such that $P = 5P_0$. Give your answer in cm.

T2-C [4.5 pts]. Caustics and Cusp

T2-C1 [0.5 pts]. Consider a ray A that strikes the mirror at an angle θ, as shown in Figure 2g. Upon reflection, the equation corresponding to this ray in the coordinate system defined in Figure 2g. is $y = m_A x + b_A$. Write m_A and b_A in terms of θ and R

Hint: the expected form of $m_A = K_1 \cot(K_2\theta)$ and $b_A = R\frac{K_3}{\cos(K_4\theta)}$, with K_i numerical constants.

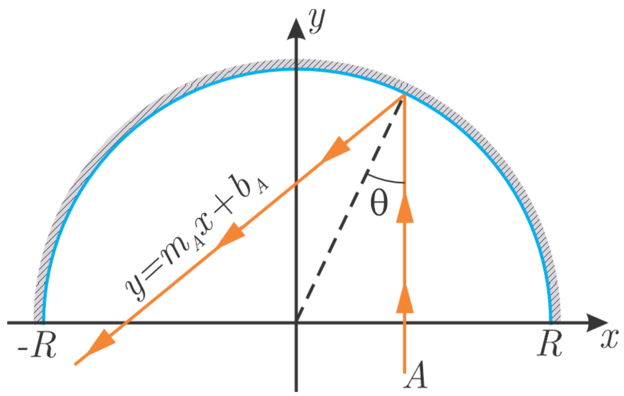

Fig. 2g.

T2-C2 [2.0 pts]. Now consider a ray B, parallel to A, that strikes the surface at an angle $\theta + \Delta\theta$, with $\Delta\theta \ll \theta$. Upon reflection, the equation corresponding to this ray is $y = m_B x + b_B$. Write m_B and b_B in terms of θ, $\Delta\theta$, and R, approximating to first order in $\Delta\theta$.

T2-C3 [1.0 pts]. Find the coordinates of the point of intersection (X_c, Y_c) between the reflected rays of A and B. Write your answer in terms of R and θ.

T2-C4 [1.0 pts]. For $\theta \ll 1$, we have $Y_c = v|X_c|^{p/q} + u$. Write v and u in terms of R and find the values of the integers p and q (equivalently, give the rational number p/q).