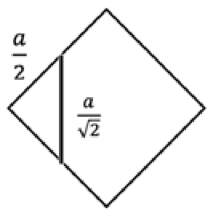
# SOLUTION T1 [10 pts]. Three problems

## T1-A [3.0 pts]. Hydrostatic gate

### T1-A1 [3.0 pts].

Since the dam is perfectly sealed, the area of the opening is $a \cdot (a/\sqrt{2})$:




The horizontal force exerted by the water on either side is equal to the pressure at the centroid of the cube sections on the corresponding side times the effective area, as calculated previously:

$$\Delta F = \frac{a^2}{\sqrt{2}} \rho_0 g \Delta h$$

With an effective application point at the centroid of the left region of the block. The other relevant forces are the block's weight $\rho \times g \times vol$, the vertical component of the force exerted by the fluid (buoyant force) $\rho_0 \times g \times vol$, and the force that the hinge applies to the block (which exerts no torque).
At the critical $\Delta h$, the bottom edge of the opening applies zero force to the block. For the net torque to be 0,

$$\frac{a^2}{\sqrt{2}} \rho_0 g \Delta h \left( \frac{a}{2\sqrt{2}} \right) = (\rho - \rho_0) a^3 g \left( \frac{a}{2\sqrt{2}} \right)$$

**Note:** Arms are $a/(2\sqrt{2})$ from O either in the horizontal or vertical direction for pressure difference and effective weight respectively.

$$a = \frac{\rho_0}{\sqrt{2}(\rho - \rho_0)} \Delta h$$

For $\rho = 3\rho_0$,

$$\boxed{a = \frac{\Delta h}{2\sqrt{2}} = 0.50 m}$$

## T1-B [2.5 pts]. Electron-positron pair

From the definition of angular momentum,

$$m v_0 (50 a_0) = \mu \hbar$$

$$v_0 = \frac{\mu \hbar}{50 m a_0}$$

The total energy of the system is

$$2\left(\frac{1}{2} m v_0^2\right) - \frac{k e^2}{100 a_0} = \frac{k e^2}{a_0}\left(\frac{\mu^2}{50^2} - \frac{1}{100}\right)$$

### T1-B1 [1.0 pts].

For $\mu = 4$,

$$E = -\frac{9 k e^2}{2500 a_0}$$

**SOLUTION 1:**
At the critical separation distances, $\vec{v} \perp \vec{r}$ and $L = mvr$. The energy at $r_{max}$ and $r_{min}$ can be written as

$$E = 2\frac{L^2}{2mr^2} - \frac{ke^2}{r}$$

Using $L = 2\mu\hbar = 8\hbar$ and the conservation of energy,

$$-\frac{9ke^2}{2500 a_0} = \frac{64\hbar^2}{mr^2} - \frac{ke^2}{r}$$

$$\frac{ke^2}{a_0}\left(\frac{9}{2500} r^2 - a_0 r + 64 a_0^2\right) = 0$$

$$r = \frac{a_0 \pm \sqrt{a_0^2 - \frac{576}{625} a_0^2}}{9/1250}$$

$$r = \frac{1250}{9} a_0 \left(1 \pm \frac{7}{25}\right)$$

The maximum separation corresponds to the solution with the plus sign,

$$\boxed{r_{max} = \frac{1600}{9} a_0}$$

**SOLUTION 2:**
Since the energy is negative and the electrostatic force is central and proportional to $r^{-2}$, the trajectory of the particles will be elliptical with respect to the CM. The energy can be written in terms of the semi-major axis $x$ as

$$E = -\frac{ke^2}{2x}$$

$$x = \frac{1250}{9}a_0$$

Since $\vec{v}_0$ is perpendicular to the vector that connects the two particles, the initial position must correspond to either $r_{max}$ or $r_{min}$. The other critical distance, $r_2$ is given by the definition of $x$:

$$2x = r_0 + r_2$$

$$\boxed{r_2 = \left(\frac{2500}{9} - 100\right)a_0 = \frac{1600}{9}a_0}$$

Which is greater than $r_0$ and corresponds to $r_{max}$

**SOLUTION 3:**
The general equation for the trajectory in polar coordinates is

$$r(\theta) = \frac{\ell}{1 + \varepsilon\cos(\theta - \theta_0)}$$

with $\ell = \frac{2L^2}{mke^2}$ the semi-latus rectum and $\varepsilon$ the eccentricity, as given by the hint in question T1-B2. The maximum separation distance is reached when $\cos(\theta - \theta_0) = -1$:

$$r_{max} = \frac{\ell}{1 - \varepsilon}$$

Calculating the values of $\ell$ and $\varepsilon$,

$$\ell = \frac{2(8\hbar)^2}{mke^2} = 128a_0$$

$$\varepsilon = \sqrt{1 - \frac{36ke^2(8\hbar)^2}{2500a_0m(ke^2)^2}}$$

$$\varepsilon = \sqrt{1 - \frac{2304}{2500}} = \frac{7}{25}$$

Replacing these values in the expression for $r_{max}$,

$$\boxed{r_{max} = \frac{128}{1-(7/25)}a_0 = \frac{1600}{9}a_0}$$

**T1-B2 [2.5 pts].**
For $\mu = 15/2$,

$$E = +\frac{ke^2}{80a_0}$$

The distance between particles follows

$$r(\theta) = \frac{\ell}{1 + \varepsilon\cos(\theta - \theta_0)}$$

Since the initial position corresponds to the closest approach, it is convenient to measure the angle $\theta$ with respect to the y axis so that $\cos\theta_0 = 1$, $\theta_0 = 0$. For $r \to \infty$,

$$1 + \varepsilon\cos\theta' \to 0$$

$$\theta' = \arccos(-1/\varepsilon)$$

Calculating the eccentricity,

$$\varepsilon = \sqrt{1 + \frac{4ke^2(15\hbar^2)}{80a_0m(ke^2)^2}}$$

$$\varepsilon = \sqrt{1 + \frac{45}{4}} = \frac{7}{2}$$

Replacing this value in the expression for $\theta$,

$$\theta' = \arccos(-2/7) = 106.60^\circ$$

Since this is the angle with respect to the $y$-axis but the initial $e^+$ velocity is directed towards the positive $x$-axis, heading downward

$$\boxed{\theta = 90^\circ - \theta' = -16.60^\circ}$$

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## T1-C [3.5 pts]. Photodissociation of ozone
**T1-C1 [2.5 pts].**

**SOLUTION 1: pure algebra**

The momentum conservation is

$$\hbar\frac{\omega}{c}\hat{k} = \vec{p} + \vec{q},$$

where $\vec{p}$ and $\vec{q}$ are the momentum of the $O_2$ and the O, respectively. Solving for $q^2$,

$$q^2 = \left(\frac{\hbar\omega}{c} - p\cos(\theta)\right)^2 + p^2\sin^2(\theta).$$

The energy conservation reads

$$\hbar\omega = \Delta U + \frac{p^2}{4m} + \frac{q^2}{2m}$$

where $\Delta U = U_f - U_i$.

For a given $\theta$, $p$ can be written in terms of $\omega$. Replacing $q^2$ on energy balance and simplifying, we get

$$p^2 - \frac{4\hbar\omega\cos(\theta)}{3c}p + \frac{2\hbar^2\omega^2}{3c^2} + \frac{4}{3}m(\Delta U - \hbar\omega) = 0.$$

This gives a quadratic equation on $p$. For $p$ to be real, the discriminant must be positive.

$$\frac{16\hbar^2\omega^2\cos^2(\theta)}{9c^2} - 4\left(\frac{2\hbar^2\omega^2}{3c^2} + \frac{4}{3}m(\Delta U - \hbar\omega)\right) \geq 0$$

The previous bound determines $\omega_{\min}$ as the minimal root of the quadratic equation on $\omega$

$$(\cos(2\theta) - 2)\hbar^2\omega^2 + 6mc^2\hbar\omega - 6\Delta U mc^2 = 0$$

$$\omega_{\pm} = \frac{-6mc^2\hbar \pm \sqrt{D}}{2\left(\cos(2\theta) - 2\right)\hbar^2}$$

$$D = 36\hbar^2m^2c^4 + 24\Delta U mc^2\hbar^2\left(\cos(2\theta) - 2\right)$$

The smallest root $\omega_+$. Thus, simplifying, one gets

$$\boxed{\omega_{\min} = \frac{3mc^2 - \sqrt{3mc^2\left(3mc^2 + 2\Delta U(\cos(2\theta) - 2)\right)}}{\hbar(2 - \cos(2\theta))}}$$

**SOLUTION 2: Center of mass approach**

*Part I: Conservation laws*

First, find the velocity of the center of mass. We get

$$(3mv_{CM}) = p_\gamma = \hbar\omega/c$$

We will use $p_\gamma$ for calculations

$$v_{CM} = p_\gamma/(3m)$$

We want to relate the relative velocities in CM. We have by magnitude

$$2mv_1 = mv_2$$

And $v_2$ is the single oxygen atom (the one we don't want). The energy conservation equation is

$$\begin{aligned} E_\gamma = p_\gamma c &= \frac{1}{2}(3m)v_{CM}^2 + \Delta U + \frac{1}{2}mv_2^2 + \frac{1}{2}(2m)v_1^2 \\ &= \Delta U + \frac{p_\gamma^2}{6m} + \frac{1}{2}\times 6m(v_1)^2 \end{aligned}$$

We can now find $v_1$ in terms of $p_\gamma$.

*Part 2: geometry and critical condition*

We see that the set of possible velocities $\vec{v}_1$ form a circle (they have constant $v_1$). We need to consider that $\theta$ is the angle between $\vec{v}_{CM}$ and $\vec{v}_{CM} + \vec{v}_1$ (this is the full velocity of particle 1).

If the circle of radius $v_1$ is too small there is no solution (the straight line at the angle will not intersect the circle). If the radius $v_1$ is big enough, there will always be two solutions intersecting the circle. The critical case (crossover from no solution to two) is when the circle is tangent to the straight line. This is the case we need.

In that case the triangle formed by $0, \vec{v}_{CM}, \vec{v}_{CM} + \vec{v}_1$ is a right triangle.

**Warning: This can only happen if $\theta \leq 90^o$. The other case will be dealt with separately.**

This way

$$v_1 = v_{CM}\sin(\theta)$$

Replace above to find that in conservation of energy

$$p_\gamma c = \Delta U + \frac{p_\gamma^2}{6m} + \frac{1}{2}\times\frac{p_\gamma^2\sin^2(\theta)}{3m}$$

now we can solve the quadratic equation for $p_\gamma$. We get (*we need to choose the correct solution of the quadratic equation*)

$$p_\gamma = \frac{3cm - \sqrt{3}\sqrt{m}\sqrt{3c^2m + 2\Delta U\cos(2\theta) - 4\Delta U}}{2\sin^2(\theta) + 1}$$

**Note:** Any of the following expressions are the same:

$$2\sin^2(\theta) + 1 = 3 - 2\cos^2(\theta) = 2 - \cos(2\theta)$$

Rewrite as the angular frequency

$$\boxed{\omega_{min} = 3mc^2\left[\frac{1-\sqrt{1-\frac{\Delta U}{3mc^2}(2\sin^2(\theta)+1)}}{(2\sin^2(\theta)+1)\hbar}\right]}$$

**Special case:** $\theta \geq 90^0$, *this is essentially impossible to see in a brute force algebraic solution, but it is an important feature of the problem. Needs a small amount of points in this solution*

For angle greater than $90^0$, as soon as the circle reaches the answer for $90^0$ it becomes big enough to allow all angles. Formula is the same with $\theta = 90^0$.

**T1-C2 [1.0 pts].**

Up to second order in $\Delta U$

$$\omega_{\rm min} = \frac{\Delta U}{\hbar} - \frac{(\cos(2\theta) - 2)\Delta U^2}{6\hbar mc^2}$$

$$\hbar\omega_{\rm min} - \Delta U = \frac{(2 - \cos(2\theta))}{6mc^2}(\Delta U)^2$$

$$\boxed{\hbar\omega_{\rm min} - \Delta U = 2.03 \times 10^{-11}\text{eV}}$$