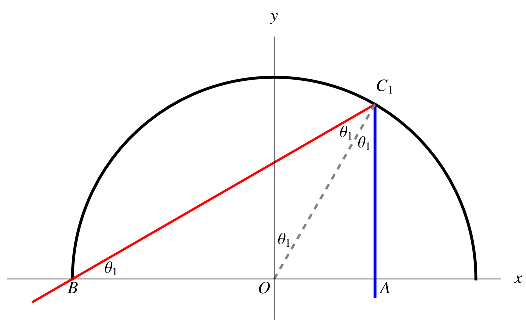
SOLUTION T2 [10 pts]. Cooking with sunlight?

PART A [1.5 pts]. Multiple reflections

T2-A1 [1.5 pts]. For $n = 1$, finding x_1 corresponds to considering the limiting case of the ray, shown in blue in the figure below.



In this case, the incident ray passes through point A, and the reflected ray, shown in red in the figure, passes exactly through point B on the edge of the mirror.
Let θ_1 be the angle formed by the radius passing through the reflection point C_1 with the vertical axis (the y axis). Let O be the center of the mirror.
Since the incident ray is parallel to the y axis, the angle of incidence is

$$\angle OC_1A = \theta_1 \,.$$

By the law of reflection,

$$\angle OC_1A = \angle OC_1B = \theta_1 \,.$$

Additionally, note that the triangle C_1BO is isosceles, so

$$\angle C_1BO = \theta_1 \,.$$

Finally, summing the internal angles of the triangle C_1BA, we have

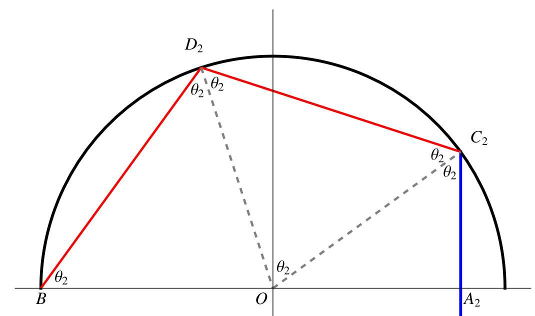
$$3\theta_1 + \frac{\pi}{2} = \pi,$$

$$\theta_1 = \frac{\pi}{6}.$$

Correspondingly, from the triangle OC_1A, we obtain

$$\boxed{x_1 = R\sin\frac{\pi}{6} = \frac{R}{2}.}$$

For $n = 2$, the limiting case for two reflections is shown in the figure below.



In analogy with the case $n = 1$, we let θ_2 be the angle between the radius passing through the reflection point C_2 and the mirror axis. We see that the trajectory of the reflected ray determines two isosceles triangles with a common side and a base angle of θ_2. Also note that the ray's trajectory forms half of a regular pentagon. Summing over the internal angles of the two isosceles triangles and the rectangular triangle, we obtain

$$5\theta_2 + \frac{\pi}{2} + \pi = 3\pi,$$

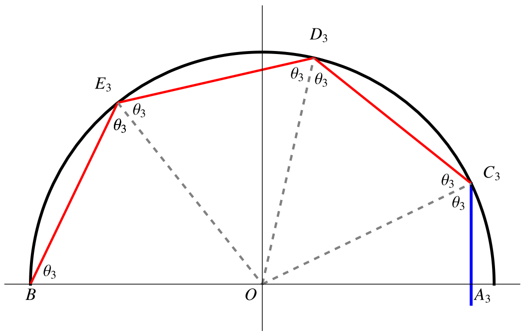
which simplifies to

$$\theta_2 = \frac{3\pi}{10}.$$

Correspondingly,

$$\boxed{x_2 = R\sin\frac{3\pi}{10} = \frac{1+\sqrt{5}}{4}R \approx 0.8R.}$$

Case n=3: For completeness, we present the limiting case for 3 reflections in the figure below.



Letting θ_3 be the incidence angle, we observe that the trajectory of the reflected ray determines three isosceles triangles with common sides and a base angle of θ_3. Also notice that the ray's trajectory now forms half of a regular heptagon. Summing over the internal angles of the three isosceles triangles and the rectangular triangle, we obtain

$$7\theta_3 + \frac{\pi}{2} + \pi = 4\pi,$$

which simplifies to

$$\theta_3 = \frac{5\pi}{14}$$

Correspondingly

$$\boxed{x_3 = R\sin\frac{5\pi}{14} \approx 0.9R.}$$

Generalizing the result, the limiting case for n reflections corresponds to the ray with incidence angle θ_N.
The trajectory of the reflected ray determines n isosceles triangles with common sides and a base angle of θ_N.
Also note that the ray's trajectory now forms half of a regular $2n+1$-gon. Summing over the internal angles of the n isosceles triangles and the rectangular triangle determined by the incident ray, we obtain

$$(2n\theta_N + \pi) + \left(\frac{\pi}{2} + \theta_N\right) = (n+1)\pi,$$

$$(2n+1)\theta_N + \frac{3\pi}{2} = (n+1)\pi,$$

which simplifies to

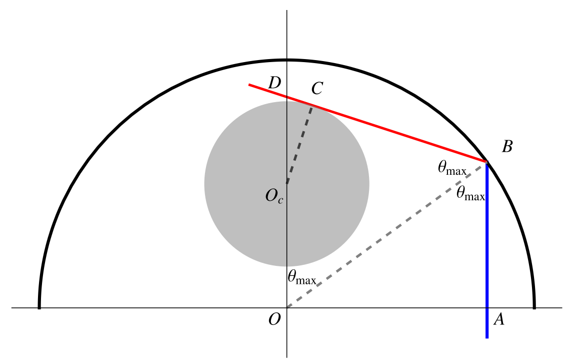
$$\theta_N = \frac{2n-1}{4n+2}\pi$$

Correspondingly

$$\boxed{x_n = R\sin\frac{2N-1}{4N+2}\pi = R\cos\frac{\pi}{2N+1}}$$

T2-B [4.0 pts]. Solar Cooker

T2-B1 [2.0 pts].
The limiting case is shown in the figure below.



- O_c is the center of the container.
- A is the incidence point on the x-axis
- B is the reflection point
- The reflected ray is tangential to the container at the point C
- The reflected ray crosses the y-axis at D.

As the triangle ODB is isosceles,

$$\angle ODB = \pi - 2\theta_{\max},$$

$$\cos\theta_{\max} = \frac{R}{2\overline{OD}}.$$

$$\overline{OD} = \frac{R}{2\cos\theta_{\max}}.$$

The triangle O_cDC is rectangle

$$\begin{aligned} a &= \overline{O_cD}\sin(\pi - 2\theta_{\max}) \\ &= \left(\overline{OD} - \frac{R}{2}\right)\sin(2\theta_{\max}) \\ &= \left(\frac{R}{2\cos\theta_{\max}} - \frac{R}{2}\right)\sin(2\theta_{\max}) \\ &= R\sin\theta_{\max} - \frac{R}{2}\sin(2\theta_{\max}) \end{aligned}$$

The coefficients read

$$\boxed{\alpha = R}$$

$$\boxed{\beta = -\frac{R}{2}}$$

T2-B2 [1.5 pts]. Without the mirror:

$$P_0 = 2Ial,$$

where I is the sunlight intensity and l is the length of the cylinder.
With the mirror:

$$\begin{aligned} P &= 2I\overline{OA}l \\ &= 2IRl\sin\theta_{\max} \end{aligned}$$

Thus

$$\begin{aligned} \frac{P}{P_0} &= \frac{2IRl\sin\theta_{\max}}{2Ial} \\ &= \frac{R\sin\theta_{\max}}{R\sin\theta_{\max}(1-\cos\theta_{\max})} \end{aligned}$$

$$\boxed{\frac{P}{P_0} = \frac{1}{1-\cos\theta_{\max}}.}$$

Alternative answer in terms of α and β: Without the mirror:

$$P_0 = 2Ial = 2I(\alpha\sin\theta_{\max} + \beta\sin(2\theta_{\max}))l$$

where I is the sunlight intensity and l is the length of the cylinder.

With the mirror:

$$\begin{aligned} P &= 2I\overline{OA}l \\ &= 2IRl\sin\theta_{\max} \end{aligned}$$

Thus

$$\begin{aligned} \frac{P}{P_0} &= \frac{2IRl\sin\theta_{\max}}{2Ial} \\ &= \frac{R\sin\theta_{\max}}{\alpha\sin\theta_{\max} + \beta\sin(2\theta_{\max})} \end{aligned}$$

$$\boxed{\frac{P}{P_0} = \frac{R}{\alpha + 2\beta\cos\theta_{\max}}.}$$

T2-B3 [0.5 pts]. If $P = 5P_0$

$$5 = \frac{1}{1-\cos\theta_{\max}},$$

$$\cos\theta_{\max} = \frac{4}{5},$$

$$\sin\theta_{\max} = \frac{3}{5}.$$

Finally

$$a = (1\text{m})\frac{3}{5}\left(1 - \frac{4}{5}\right).$$

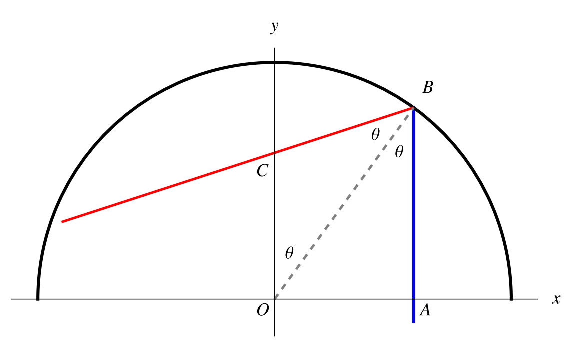
$$\boxed{a = 12\,\text{cm}.}$$

Alternative solution parametrized in terms of α and β

$$a = -\frac{R\sqrt{100\beta^2 - (R-5\alpha)^2}}{50\beta}$$

T2-C [4.5 pts]. Caustics and Cusp

T2-C1 [0.5 pts].



The reflected ray forms an angle 2θ with the vertical axis. Therefore, the slope of the line is

$$\boxed{m_{_A} = \cot(2\theta).}$$

The reflected ray intersects the y-axis at C, $\overline{OC} = b_A$, forming the isosceles triangle OBC. from the intersection

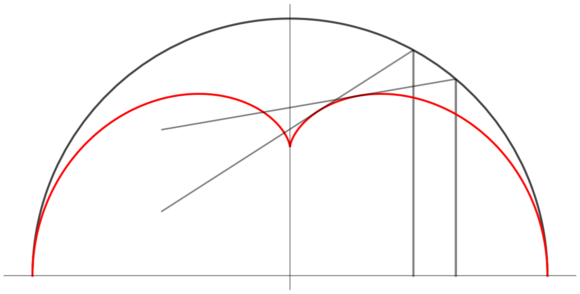
$$\cos\theta = \frac{R}{2b_A}.$$

$$\boxed{b_A = \frac{R}{2\cos\theta}.}$$

Equivalent expressions:

$$m_{_A} = \cot(2\theta) = \frac{1}{\tan(2\theta)} = \frac{\cot(\theta)}{2} - \frac{\tan(\theta)}{2}$$
$$b_A = \frac{R}{2\cos\theta} = \frac{R}{2}\sec\theta$$

T2-C2 [2.0 pts].



The slope is

$$\begin{aligned} m_{_B} &= \cot(2\theta + 2\Delta\theta) \\ &= \frac{\cot(2\theta)\cot(2\Delta\theta) - 1}{\cot(2\theta) + \cot(2\Delta\theta)} \\ &= \frac{\cot(2\theta) - \tan(2\Delta\theta)}{\cot(2\theta)\tan(2\Delta\theta) + 1} \\ &= (\cot(2\theta) - 2\Delta\theta)(1 - 2\Delta\theta\cot(2\theta)) + \dots \\ &= \cot(2\theta) - 2\Delta\theta(1 - \tan^2(2\Delta\theta)) + \dots \end{aligned}$$

Then, the first-order expansion of $m_{_B}$ reads

$$\boxed{m_{_B} = \cot(2\theta) - 2\csc^2(2\theta)\Delta\theta.}$$

On the other hand, the intercept is

$$\begin{aligned} b_{_B} &= \frac{R}{2}\sec(\theta + \Delta\theta) \\ &= \frac{R}{2}\frac{\sec(\theta)\sec(\Delta\theta)}{1 - \tan(\theta)\tan(\Delta\theta)} \\ &= \frac{R}{2}(\sec(\theta)(1 + \tan(\theta)\Delta\theta)) + \dots \end{aligned}$$

and the first-order expansion reads

$$\boxed{b_{_B} = \frac{R}{2}\sec(\theta)\left(1 + \tan(\theta)\Delta\theta\right)}$$

Equivalent expressions:

$$\begin{aligned} m_{_B} &= \cot(2\theta) - 2\csc^2(2\theta)\Delta\theta \\ &= \frac{1}{2}\csc^2(2\theta)(\sin(4\theta) - 4\Delta\theta) \\ &= \frac{1}{2}\left(\cot(\theta) - \tan(\theta) - \csc^2(\theta)\sec^2(\theta)\right)\Delta\theta \end{aligned}$$

$$\begin{aligned} b_{_B} &= \frac{R}{2}\sec(\theta)\left(1 + \tan(\theta)\Delta\theta\right) \\ &= \frac{R}{2\cos(\theta)}\left(1 + \tan(\theta)\Delta\theta\right) \end{aligned}$$

T2-C3 [1.0 pts].
The coordinates of the intersection point, (X_C, Y_C), fulfill the following equations:

$$\begin{aligned} Y_C &= m_{_A} X_C + b_{_A} \\ Y_C &= m_{_B} X_C + b_{_B}, \end{aligned}$$

Solving:

$$(X_C, Y_C) = \left(-\frac{b_{_A} - b_{_B}}{m_{_A} - m_{_B}}, \frac{m_{_A} b_{_B} - b_{_A} m_{_B}}{m_{_A} - m_{_B}}\right)$$

Replacing $m_{_B}$ and $b_{_B}$, we find:

$$(X_C, Y_C) = \left(R\sin^3(\theta), \frac{R}{2}\cos(\theta)(2 - \cos(2\theta))\right)$$

$$\boxed{X_C = R\sin^3(\theta)}$$

$$\boxed{Y_C = \frac{R}{2}\cos(\theta)(2-\cos(2\theta))}$$

Equivalent expressions:

$$\begin{aligned} Y_C &= \frac{R}{2}\cos(\theta)(2-\cos(2\theta)) \\ &= \frac{R}{4}(3\cos(\theta)-\cos(3\theta)) \\ &= \frac{R}{4}(3\cos(\theta)-\cos^3(\theta)+3\sin^2(\theta)\cos(\theta)) \end{aligned}$$

T2-C4 [1.0 pts]. For $\theta \ll 1$, we use the approximations

$$\sin\theta \approx \theta,$$

$$\cos\theta \approx 1-\frac{\theta^2}{2}.$$

Thus, near the cusp, the caustic is approximated as

$$(X_C, Y_C) \approx \left(R\theta^3, \frac{R}{4}(2+3\theta^2)\right)$$

$$(X_C, Y_C) \approx \left(R\theta^3, \frac{R}{2}+\frac{3R}{4}\theta^2\right)$$

Solving for θ, we get

$$\theta = \left(\frac{X_C}{R}\right)^{\frac{1}{3}},$$

and substitute in Y_C

$$Y_C = \frac{R}{2}+\frac{3R}{4}\left(\frac{X_C}{R}\right)^{\frac{2}{3}}$$

Comparing with

$$Y_C = v|X_C|^{p/q}+u,$$

we find

$$\boxed{v = \frac{3}{4}R^{\frac{1}{3}}}$$

$$\boxed{u = \frac{R}{2},}$$

$$\boxed{p = 2}$$

$$\boxed{q = 3}$$

This shows that the shape of the cusp is universal and that its tip is located at $y = \frac{R}{2}$.