SOLUTION T3 [10 pts]. Chashing abolute zero

T3-A [1.0 points]

T1-A1 [0.2 pts]. Field $\vec{H}$ inside the torus is almost entirely confined to its core, forming concentric circumferential circles. Therefore, from magnetic circuital law $\oint_C \vec{H}\cdot d\vec{\ell} = I_C$ we have $H\cdot 2\pi R = NI$. Dividing and multiplying by $A = \pi r^2$ yields to

$$H = \frac{NI}{2\pi R}\frac{A}{\pi r^2} = \frac{NIA}{V}.$$

T3-A2 [0.6 pts]. When I is changed, there is induced an *emf* $\mathscr{E}_{\text{ind}}$. Thus, since

$$\mathscr{E} + \mathscr{E}_{\text{ind}} = \text{resistance} \times I \approx 0,$$

from Faraday's law of induction, the work done by the external *emf* is given by

$$dW_{emf} = \mathscr{E}dq = -\mathscr{E}_{\text{ind}} I dt = \frac{d\Phi_B}{dt} I dt = I d\Phi_B,$$

where dq is the quantity of electric charge flowing during the time interval dt and $\Phi_B = NAB$ is the flux of the field $\vec{B}$ inside the torus. Therefore,

$$dW_{emf} = NIA\; dB = VH\; dB.$$

T3-A3 [0.2 pts]. Since $\vec{B}$, $\vec{H}$ and $\vec{M}$ are paralell vectors, from $\vec{B} = \mu_0\vec{H} + \mu_0\vec{M}$ we get

$$dB = \mu_0 dH + \mu_0 dM,$$

from which, after substitution in the answer of question T3-A2, yields to

$$dW_{emf} = \mu_0 VHdH + \mu_0 VHdM.$$

Hence,

$$dW = \mu_0 VHdM.$$

T3-B [3.0 pts]

T3-B1 [1.5 pts]. The first law of thermodynamics for the torus is $dU = dW + dQ = \mu_0 VHdM + CdT$. At constant M, we have $dU = C_M dT = n\lambda dT/T^2$, i. e., U depends only on T. Hence, for isothermal processes, $dQ = -dW = -\mu_0 VHdM$. Furthermore, from the equation of state we have for isothermal processes that $TVdM = nKdH$ and then

$$dQ = -\mu_0 VH\left(\frac{nK}{TV}dH\right) = -\mu_0\frac{nK}{T}HdH.$$

Integration from H_i to H_f gives

$$Q = -\mu_0\frac{nK}{T}\frac{H_f^2 - H_i^2}{2}.$$

Note: This is proportional to the volume because it has the number of moles.

T3-B2 [1.5 pts]. For adiabatic processes, the first law yields to $dU = dW$ and then

$$\frac{n\lambda}{T^2}dT = \mu_0 VHdM = \mu_0 V\frac{TMV}{nK}dM,$$

where we have used the equation of state to substitute H. Separating variables we have

$$\frac{n\lambda}{T^3}dT = \mu_0\frac{V^2}{nK}MdM,$$

and hence, after integration from (T_i, M_i) to (T_f, M_f),

$$\frac{n\lambda}{2}\left(\frac{1}{T_i^2} - \frac{1}{T_f^2}\right) = \mu_0\frac{V^2}{nK}\frac{M_f^2 - M_i^2}{2}.$$

Since $M = nKH/TV$, this equation simplifies to

$$\frac{\lambda}{T_i^2} - \frac{\lambda}{T_f^2} = \mu_0 K\left(\frac{H_f^2}{T_f^2} - \frac{H_i^2}{T_i^2}\right),$$

from which

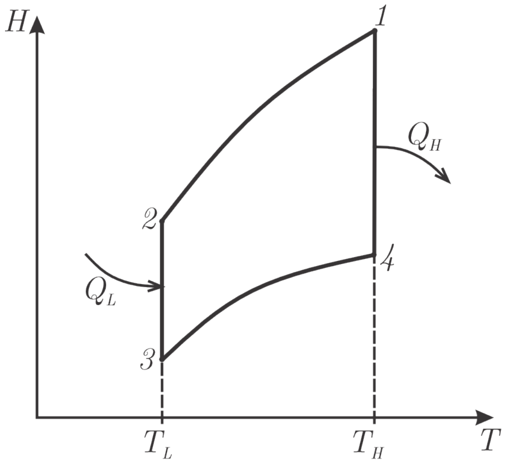
$$\frac{T_f^2}{T_i^2} = \frac{\lambda + \mu_0 KH_f^2}{\lambda + \mu_0 KH_i^2}.$$

Hence

$$\Delta T = T_f - T_i = T_i\left(\sqrt{\frac{\lambda + \mu_0 KH_f^2}{\lambda + \mu_0 KH_i^2}} - 1\right).$$

T3-C [6.0 pts]

T3-C1 [0.2 pts].



T3-C2 [1.5 pts]. In the Carnot cycle

$$\frac{Q_c}{Q_h} = \frac{T_c}{T_h}.$$

Particularly, for the Carnot cycle of paramagnetic refrigeration, we have $Q_c = |Q_{2\to3}| = Q_{2\to3}$ and $Q_h = |Q_{4\to1}| = -Q_{4\to1}$. From the answer to question T3-B1,

$$Q_c = -\mu_0 \frac{nK}{T_c}\frac{H_3^2 - H_2^2}{2} = \mu_0 \frac{nK}{T_c}\frac{H_2^2 - H_3^2}{2},$$
$$Q_h = -\left(-\mu_0 \frac{nK}{T_h}\frac{H_1^2 - H_4^2}{2}\right) = \mu_0 \frac{nK}{T_h}\frac{H_1^2 - H_4^2}{2},$$

where $T_c = T_2 = T_3$ and $T_h = T_4 = T_1$ are the respective temperatures of the isothermal processes $2 \to 3$ and $4 \to 1$. Therefore, after substitution and simplification in the Carnot identity, we have

$$\frac{T_h}{T_c}\frac{H_2^2 - H_3^2}{H_1^2 - H_4^2} = \frac{T_c}{T_h},\quad \frac{H_2^2 - H_3^2}{T_c^2} = \frac{H_1^2 - H_4^2}{T_h^2}.$$

Since, by the equation of state, $M = nKH/TV$, we get

$$M_2^2 - M_3^2 = M_1^2 - M_4^2,\ M_1 = \sqrt{M_2^2 - M_3^2 + M_4^2}.$$

T3-C3 [0.8 pts]. By substituting the given numerical values

$$Q_c = \mu_0 \frac{nK}{T_c}\frac{H_2^2 - H_3^2}{2} = 1.29 \times 10^{-1}\ \text{J}.$$

Hence

$$|\Delta T|_{\textbf{helium}} = \frac{1.29 \times 10^{-1}\ \text{J}}{130\ \text{kg}\cdot\text{m}^{-3} \times 10^{-3}\ \text{m}^3 \times 10^2\ \text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}}$$
$$= \frac{1.29}{1.30} \times 10^{-2}\ \text{K} = 9.92 \times 10^{-3}\ \text{K}.$$

Hence

$$T_f = T_i - |\Delta T| = 1.00\ \text{K} - 0.00992\ \text{K} = 0.99008\ \text{K}.$$

T3-C4 [2.0 pts]. From the Carnot cycle identity $dQ_c/dQ_h = T_c/T_h$ and the first law in differential form $dQ_h = dW + dQ_c$, we have

$$\frac{dW + dQ_c}{dQ_c} = \frac{dW}{dQ_c} + 1 = -\frac{Pdt}{C_c dT_c} + 1 = \frac{T_h}{T_c},$$

where $dW = Pdt$ and $dQ_c = -C_c dT_c$ have been used. Hence, after separation of variables,

$$dt = -\frac{C_c T_h}{P}\left(\frac{dT_c}{T_c} - \frac{dT_c}{T_h}\right).$$

Integrating from $t = 0$ ($T_c = T_0$) to t ($T_c = T$) we get

$$t = \frac{C_c T_h}{P}\left(\ln\frac{T_0}{T} - \frac{T_0 - T}{T_h}\right).$$

T3-C5 [1.5 pts]. Since $W = Pt$ and $Q_c = C_c(T_0 - T)$,

$$W = Pt = C_c T_h \ln\frac{T_0}{T} - Q_c.$$

Therefore

$$\text{COP} = \frac{Q_c}{W} = \left(\frac{T_h}{T_0 - T}\ln\frac{T_0}{T} - 1\right)^{-1}.$$